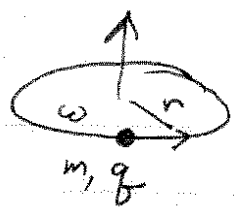
Physics 492 - Quantum II

Lecture 10: Spin angular momentum and magnetic coupling

Magnetic moment of electrons

Because electrons move the produce currents. For example, consider the simplest classical circular orbit

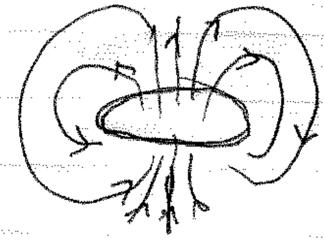


Current $I = \frac{q}{T} = \frac{\omega}{2\pi} q$

The angular momentum $L = m\omega r^2$

$$\Rightarrow \quad I = \frac{q}{2\pi r^2 m} L$$

Associated with this current loop is a magnetic field



This is ~~the~~ field is of the dipole form

A magnetic dipole moment for a loop of area A is $\vec{\mu} = \underbrace{\left[\frac{\mu_0}{4\pi}\right]}_{=1 \text{ in cgs}} \frac{I}{c} A \, \vec{n}$ ← normal

$\begin{pmatrix} \text{For the charge on trajectory} \\ \text{in a loop } A = \pi r^2 \end{pmatrix} \Rightarrow \vec{\mu} = \frac{q}{2mc} L \, \vec{n}$

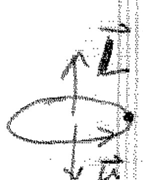
$$\Rightarrow \boxed{\vec{\mu} = \frac{q}{2mc}\vec{L} = \gamma \vec{L} \qquad \gamma = \text{"gyromagnetic ratio"}}$$

(cgs units)

General expression

Quantum description for electron

$$\hat{\vec{\mu}} = -\frac{e}{2m}\hat{\vec{L}} = -\frac{e\hbar}{2mc}\hat{\vec{l}}$$



(negative charge)

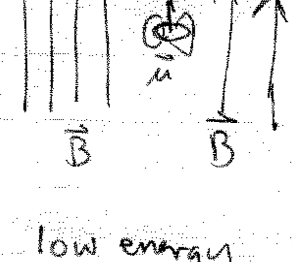
$$\boxed{\hat{\vec{\mu}} = -\mu_B \hat{\vec{l}}, \quad \mu_B = \frac{e\hbar}{2mc} \qquad \mu_B = 0.93 \times 10^{-20} \frac{\text{ergs}}{\text{Gaus}} = \text{Bohr magneton}}$$

Interaction with external $\vec{B}$-field

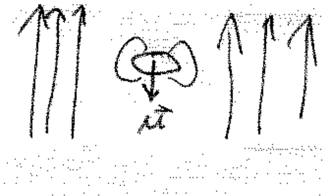
Because electrons with orbital angular momentum produce currents, they will interact with externally applied magnetic fields. The interaction energy is

$$H_{int} = -\vec{\mu} \cdot \vec{B}$$

(Try to align $\vec{\mu}$ with $\vec{B}$)



low energy



high energy

<u>Zeeman effect</u>: Splitting of spectral lines in an external magnetic field

$$\hat{H} = \hat{H}_0 + \underbrace{\hat{H}_{int}}_{\text{"perturbation"}}$$

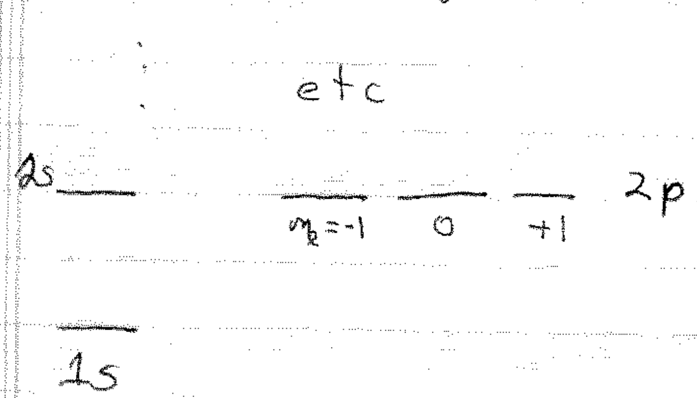
Hydrogen $\hat{H}_0 = \frac{\hat{P}_r^2}{2m} + \frac{\hat{L}^2}{2mr^2} - \frac{e^2}{r} = \sum_{n,l,m}' E_n \, |n l m_l\rangle\langle n l m_l|$

(for bound part of spectrum $E_n = \frac{-R}{n^2}$)

$\hat{H}_{int} = + \frac{\mu_B}{\hbar} \hat{\vec{L}} \cdot \vec{B}$ Take $\vec{B} = B\vec{e}_z$ (choose z-axis along $\vec{B}$)

$\Rightarrow \hat{H}_{int} = + \left(\frac{\mu_B B_z}{\hbar}\right) \hat{L}_z$

Without $\vec{B}$-field



(Degeneracy for $2l+1$ sub-states for any l)
essential; spherical symmetry

(Recall: For hydrogen, also degeneracy for different l with same principle quantum number ('accidental'))

Now, adding $\vec{B}$-field, the system is no longer spherical symmetric ($\vec{B}$ field along one axis).

However, the eigenvectors of $\hat{H}_0$, $|n, \ell, m_\ell\rangle$ are still eigenvectors of $\hat{L}_z$

$$\hat{L}_z |n, \ell, m_\ell\rangle = \hbar m_\ell |n, \ell, m_\ell\rangle$$

$\Rightarrow$ Magnetic interaction

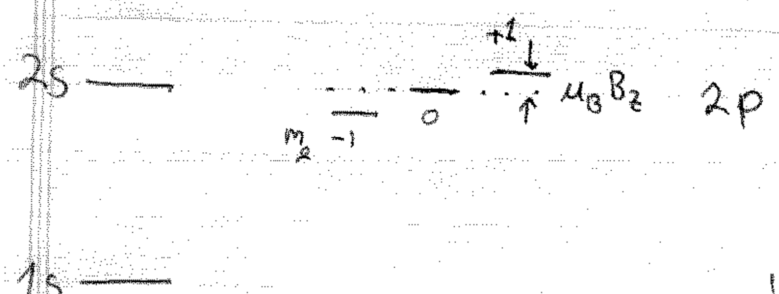
$$\hat{H}_{int} |n, \ell, m_\ell\rangle = + m_\ell \underbrace{(\mu_B B_z)}_{\text{energy} \ll \frac{e^2}{a_0} = 13.6\,eV}$$

example: Earth's magnetic field $B \sim 0.5$ Gauss

$$\Rightarrow \mu_B B_{earth} \approx 0.5 \times 10^{-20}\,erg \cong 3 \times 10^{-9}\,eV$$

Tiny

New energy levels: $\hat{H} |n \ell m_\ell\rangle = \left[-\frac{R}{n^2} + m_\ell (\mu_B B_z) \right] |n, \ell, m_\ell\rangle$

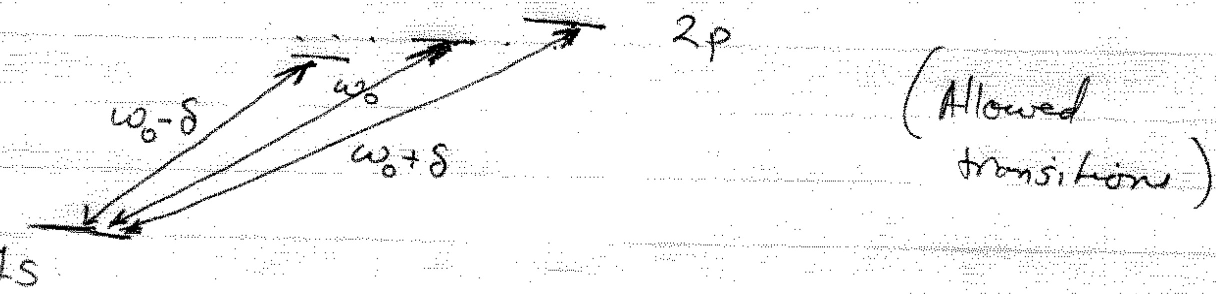


(degeneracy broken by perturbation)

"Zeeman shift"

$m_\ell \equiv$ "magnetic quantum number"

Spectral lines



Single spectral line without $\vec{B}$ $\Rightarrow$ ~~One~~ Three spectral lines with $\vec{B}$

"Normal Zeeman effect"

Historically, the Zeeman effect was observed in the early 20^{th} century, soon after the discovery of the electron, but before quantum theory. Lorentz explained it classically, one of the first applications of the electron theory, for which he and Peter Zeeman shared the nobel prize. However, it was already clear that this wasn't the whole story. For alkali atoms (e.g. sodium) the spectrum was split into more than three lines, something unexplainable by the classical theory. This was known as the "anomalous Zeeman effect". Landé fit this measured spectrum with a "fudge factor" $\mu = g \frac{\mu_B}{\hbar}$ (g = Landé "g factor")

and half-integer m values in some cases.

Pauli, introduced a additional "degree of freedom" in order to account for the periodic table via his exclusion principle.

In 1924 Uhlenbeck and Goudsmit explained this and the Zeeman effect (as well as fine structure, to be discussed later in the semester) through the introduction of spin angular momentum.

The electron spin has associated with it an intrinsic magnetic moment.

$$\hat{\vec{\mu}}_{\substack{\text{spin}\\ \text{electron}}} = -g_s \frac{\mu_B}{\hbar} \hat{\vec{S}}$$

Where $\hat{\vec{S}}$ = spin angular momentum operator (with $s = 1/2$ for electron)

Empirically $g_s = 2$ (plus QED correction)

The derivation of the g-factor for electrons waited for the Dirac equation which gave a relativistic description of the electron.

Pauli Matricies:

$$\vec{\mu}_s = -2\frac{\mu_B}{\hbar}\hat{\vec{S}} \equiv -\mu_B \hat{\vec{\sigma}}$$

where $\hat{\vec{S}} = \frac{\hbar}{2}\hat{\vec{\sigma}}$

$\hat{\vec{\sigma}}$ = Pauli operator

In standard basis $\{ |\uparrow\rangle = |\tfrac{1}{2}, m_s = \tfrac{1}{2}\rangle$
$|\downarrow\rangle = |\tfrac{1}{2}, m_s = -\tfrac{1}{2}\rangle \}$

we can use the $j = 1/2$ angular momentum matricies we found in the last lecture:

$$\hat{\sigma}_x \doteq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \hat{\sigma}_y \doteq \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \hat{\sigma}_z \doteq \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

These are known as the <u>Pauli matricies</u>. They are important in the theory of spin. As well they have useful algebraic properties which apply to any Hilbert space spanned by two orthogonal vectors.

<u>Note:</u> One usually defines $\hat{\sigma}_+ = |\uparrow\rangle\langle\downarrow|$

$\hat{\sigma}_- = |\downarrow\rangle\langle\uparrow|$

$$\Rightarrow \hat{\sigma}_+ \doteq \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \frac{\hat{S}_+}{\hbar} \qquad \hat{\sigma}_- \doteq \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \frac{\hat{S}_-}{\hbar}$$

Properties of the Pauli matricies

Eigenvalues: Since the operator $\vec{e}_n \cdot \hat{\vec{S}}$ has eigenvalues $\pm\frac{\hbar}{2}$, the Pauli matricies have eigenvalues ± 1

$$\hat{\sigma}_z |\uparrow\rangle = +1 \; |\uparrow\rangle \qquad \hat{\sigma}_z |\downarrow\rangle = -1 \; |\downarrow\rangle$$

$$\hat{\sigma}_z \doteq \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \hat{\sigma}_z^2 \doteq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \hat{\mathbb{1}}$$

Since $\hat{\mathbb{1}}$ is the same in all basis $\hat{\sigma}_z^2 = \hat{\mathbb{1}}$

Thus, for any direction $\boxed{\hat{\sigma}_i^2 = \hat{\mathbb{1}}}$

Check $\hat{\sigma}_x^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

$\hat{\sigma}_y^2 = \begin{bmatrix} 0 & -i \\ +i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ +i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

Trace: The trace of a matrix is the sum of its diagonal elements.

$$\mathrm{Tr}(\hat{A}) = \sum_n \langle n | \hat{A} | n \rangle = \sum_n A_{nn}$$

This sum is basis independent

$\Rightarrow$ ~~Tr~~ $\mathrm{Tr}(\hat{A})$ = Sum eigenvalues $\Rightarrow \boxed{\mathrm{Tr}(\hat{\sigma}_i) = 0}$