Lecture 10: Spin angular momentum and magnetic coupling

Magnetic moment of electrons

Because electrons move they produce currents. For example, consider the simplest classical circular orbit

\[ I = \frac{q}{r} = \frac{\omega q}{2\pi} \]

The angular momentum \( L = m\omega r^2 \)

\[ \Rightarrow I = \frac{q}{2\pi r^2 m} L \]

Associated with this current loop is a magnetic field

\[ \vec{H} = \frac{I}{4\pi} \hat{A} \]

This is the field is of the dipole form

A magnetic dipole moment for a loop of area \( A \) is

\[ \vec{m} = \left[ \frac{\mu_0}{4\pi} \right] \hat{A} \quad \text{normal} \]

\( = 1 \) in cgs

(For the charge on trajectory) \( \Rightarrow \vec{m} = qL \hat{\mathbf{r}} \)

(in a loop \( A = \pi r^2 \) \( \frac{q}{2mc} \))
Quantum description for electron

\[ \hat{\mu} = -\frac{e}{2m} \hat{L} = -\frac{e\hbar}{2mc} \hat{L} \]

\[ \hat{\mu} = -\mu_B \hat{L} \]

\[ \mu_B = \frac{e\hbar}{2mc} \]

\[ \mu_B = 0.93 \times 10^{-20} \text{ ergs Gauss} = \text{Bohr magneton} \]

Interaction with external \( \vec{B} \)-field

Because electrons with orbital angular momentum produce currents, they will interact with externally applied magnetic fields. The interaction energy is

\[ H_{\text{int}} = -\vec{\mu} \cdot \vec{B} \]

\( \left( \text{Try to align} \right) \)

\[ \vec{B} \quad \vec{B} \]

\[ \text{low energy} \quad \text{high energy} \]
Zeeman effect. Splitting of spectral lines in an external magnetic field.

\[ \hat{H} = \hat{H}_0 + \hat{H}_{\text{int}} \]  

"perturbation"

Hydrogen:

\[ \hat{H}_0 = \frac{\hat{p}_r^2}{2m} + \frac{\hat{L}^2}{2\hbar^2} - \frac{e^2}{4\pi \varepsilon_0 r} = \sum_{n, l, m} E_n \hat{\psi}_{n,l,m} \]

(for bound part of spectrum \( E_n = \frac{-\hbar^2}{2m} \))

\[ \hat{H}_{\text{int}} = + \frac{\mu_B}{\hbar} \hat{L} \cdot \vec{B} \]

Take \( \vec{B} = B \vec{e}_z \)

(choose \( z \)-axis along \( \vec{B} \))

\[ \Rightarrow \hat{H}_{\text{int}} = + \left( \frac{\mu_B B}{\hbar} \right) \hat{L}_z \]

Without \( \vec{B} \)-field

\[ \text{etc} \]

As \( n = 1 \) \( l = 0 \) \( m = 0 \)

\[ \text{1s} \]

(Recall: For hydrogen, also degeneracy for different \( l \) with same principle quantum number (accidental'))

\[ \text{(Degeneracy for 2l+1 sub-states for any } l \text{)} \]

\[ \text{essential spherical symmetry} \]
Now, adding B-field, the system is no longer spherical symmetric (B field along one axis).

However, the eigenvectors \( |n, l, m, m_e> \) are still eigenvectors of \( \hat{L}_z \)

\[
\hat{L}_z |n, l, m, m_e> = i m_e |n, l, m, m_e>
\]

\( \Rightarrow \) Magnetic interaction

\[
\hat{A}_{int} |n, l, m, m_e> = -m_e \left( \mu_B B_z \right) \frac{\hbar}{2} \text{energy} \ll \frac{e}{a_o} = 13.6 \text{eV}
\]

Example: Earth's magnetic field \( B \sim 0.5 \text{ Gauss} \)

\( \Rightarrow \mu_B B_{Earth} \sim 0.5 \times 10^{-20} \text{ erg} \sim 3 \times 10^{-4} \text{ eV} \) (Tiny)

New energy levels:

\[
\hat{H}_l |n, l, m, m_e> = \left[ -\frac{R}{\hbar^2} + m_e \left( \mu_B B_z \right) \right] |n, l, m, m_e>
\]

\( 2s \) \( \xrightarrow{\text{Zeeman shift}} \) \( \frac{+1}{2} \mu_B B_z \) \( 2p \) (degeneracy broken by perturbation)

\( 1s \) \( m_z \equiv \text{magnetic quantum number} \)
Spectral lines

\[ \alpha_0 - \delta \quad \alpha_0 \quad \alpha_2 + \delta \]

\[ 2p \]

(Allowed transition)

1s

Single spectral line → Three spectral lines with B

"Normal Zeeman effect"

Historically, the Zeeman effect was observed in the early 20th century, soon after the discovery of the electron, but before quantum theory. Lorentz explained it classically, one of the first applications of the electron theory, for which he and Peter Zeeman shared the Nobel prize. However, it was already clear that this wasn't the whole story. For alkaline atoms (e.g., sodium), the spectrum was split into more than three lines, something unexplainable by the classical theory. This was known as the "anomalous Zeeman effect". Landé fitted this measured spectrum with a "fudge factor"

\[ \mu = g \frac{eB}{h} \]

\((g = \text{Landé "g factor"})\)

and half-integer \(m\) values in some cases.
Pauli introduced an additional "degree of freedom" in order to account for the periodic table via his exclusion principle.

In 1924 Uhlenbeck and Goudsmit explained this and the Zeeman effect (as well as fine structure, to be discussed later in the semester) through the introduction of spin angular momentum.

The electron spin has associated with it an intrinsic magnetic moment.

\[ \hat{\mu}_{\text{spin, electron}} = -g_s \frac{\mu_B}{\hbar} \frac{\hat{\mathbf{s}}}{S} \]

Where \( \hat{\mathbf{s}} = \text{spin angular momentum operator} \) (with \( S = \frac{1}{2} \) for electron).

Empirically \( g_s = 2 \) (plus QED corrections).

The derivation of the g-factor for electrons waited for the Dirac equation, which gave a relativistic description of the electron.
Pauli Matrices:
\[ \hat{\mu}_s = -2 \mu_B \frac{\vec{s}}{\hbar} \]
\[ \vec{s} = \frac{\hbar}{2} \vec{\sigma} \]
where \[ \vec{\sigma} = \text{Pauli operator} \]

In standard basis \[ \uparrow \uparrow = \frac{1}{2}, m_s = \frac{1}{2} \]
\[ \downarrow \uparrow = \frac{1}{2}, m_s = -\frac{1}{2} \]
we can use the \( j = \frac{1}{2} \) angular momentum matrices we found in the last lecture:
\[ \hat{\sigma}_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \hat{\sigma}_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \hat{\sigma}_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \]

These are known as the Pauli matrices. They are important in the theory of spin. As well, they have useful algebraic properties which apply to any Hilbert space spanned by two orthogonal vectors.

**Note:** One usually defines
\[ \hat{\sigma}_+ = |\uparrow \rangle \langle \uparrow | \]
\[ \hat{\sigma}_- = |\downarrow \rangle \langle \uparrow | \]
\[ \hat{\sigma}_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \frac{\hat{\sigma}_x}{\hbar}, \quad \hat{\sigma}_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \frac{\hat{\sigma}_y}{\hbar} \]
Properties of the Pauli matrices

Eigenvalues: Since the operator $\vec{\sigma} \cdot \vec{S}$ has eigenvalues $\pm \frac{1}{2}$, the Pauli matrices have eigenvalues $\pm 1$

$$\sigma_z |\uparrow\rangle = +1 |\uparrow\rangle \quad \sigma_z |\downarrow\rangle = -1 |\downarrow\rangle$$

$$\sigma_z^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \sigma_z^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbb{1}$$

Since $\mathbb{1}$ is the same in all basis $\sigma_z^2 = \mathbb{1}$

Thus, for any direction, $\sigma_z^2 = \mathbb{1}$

Check $\sigma_x^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$

$\sigma_y^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$

Trace: The trace of a matrix is the sum of its diagonal elements.

$$\text{Tr} (\hat{A}) = \sum_n \langle n | \hat{A} | n \rangle = \sum_n A_{nn}$$

This sum is basis independent

$$\Rightarrow \quad \text{Tr}(\sigma_z) = \text{Sum eigenvalues} = 0$$