Physics 492 - Quantum II

Lecture 11: Larmor Precession and Rotations

Dynamics in a magnetic field:

A magnetic moment wants to allign itself with an externally applied magnetic field according to the Hamiltonian

$$\hat{H} = -\hat{\vec{\mu}} \cdot \vec{B} = +\gamma_J \vec{B} \cdot \hat{\vec{J}}$$

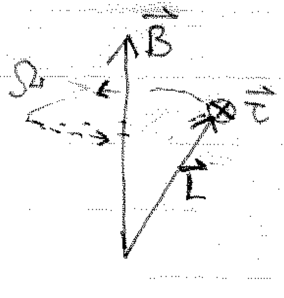
where γ_J is the gyromagnetic ratio, $\hat{\vec{\mu}} = -\gamma_J \hat{\vec{J}}$
$= g_J \mu_B / \hbar$

What is the moment is not initially alligned? How does the system evolve in time?

Classically, this is the energy of a gyroscope. The angular momentum changes due to a torque

$$\frac{d\vec{L}}{dt} = \vec{\tau} = \vec{\mu} \times \vec{B} = -\gamma_L \vec{L} \times \vec{B}$$

$$= +\vec{\Omega}_L \times \vec{L} \quad \text{where } \vec{\Omega}_L = \gamma_L \vec{B}$$



The magnetic moment precesses about the magnetic field about $\vec{B}$ at frequency $\Omega_L = \gamma B$ as a gyroscope.

This is known a Larmor precessesion

Larmor precession is also seen quantum mechanically.

Consider a magnetic moment due to orbital angular momentum only, say for $\ell=1$.

Consider an initial state $|\psi(0)\rangle = |\ell=1, m_x=1\rangle$

$$\hat{L}_x |\psi(0)\rangle = \hbar\, |\psi(0)\rangle$$

We have found $|\psi(0)\rangle = \frac{1}{2}\left(|\ell=1, m_z=1\rangle + |\ell=1, m_z=-1\rangle + \sqrt{2}\,|\ell=1, m_z=0\rangle\right)$

$\Rightarrow$ In the standard basis $|\psi(0)\rangle \doteq \frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$

The angular momentum matricies are:

$$\hat{\ell}_x \doteq \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \hat{\ell}_y \doteq \frac{i}{\sqrt{2}}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \hat{\ell}_z \doteq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$\Rightarrow$ At the initial time

$$\langle \hat{\ell}_x \rangle_{t=0} = \langle \psi(0) | \hat{\ell}_x | \psi(0) \rangle = 1$$

$$\langle \hat{\ell}_y \rangle_{t=0} = \frac{i}{4}\begin{bmatrix} 1 & \sqrt{2} & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} = 0$$

$$\langle \hat{\ell}_z \rangle_{t=0} = \frac{1}{4}\begin{bmatrix} 1 & \sqrt{2} & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} = 0$$

Since $|\psi(0)\rangle$ is not a stationary state, the expectation value of observables will evolve in time.

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle = e^{-i\vec{\Omega}\cdot\frac{\hat{\vec{L}}}{\hbar}t}|\psi(0)\rangle$$

$(\vec{\Omega} = \gamma\vec{B})$

Let us take the $\vec{B}$ field in the z-direction

$$\Rightarrow \quad \hat{H} = -\hat{\vec{\mu}}\cdot\vec{B} = +\Omega\hat{L}_z$$

$$\Rightarrow |\psi(t)\rangle = e^{-i\Omega t\,\hat{l}_z}|\psi(0)\rangle = \sum_m c_m e^{-im\Omega t}|l=1, m\rangle$$

In the standard basis $|\psi(t)\rangle \doteq \frac{1}{2}\begin{bmatrix} e^{-i\Omega t} \\ \sqrt{2} \\ e^{+i\Omega t} \end{bmatrix}$

$$\Rightarrow \langle\hat{l}_x\rangle_t = \langle\psi(t)|\hat{l}_x|\psi(t)\rangle = \frac{1}{4\sqrt{2}}\left[e^{+i\Omega t}\ \ \sqrt{2}\ \ e^{-i\Omega t}\right]\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} e^{-i\Omega t} \\ \sqrt{2} \\ e^{+i\Omega t} \end{bmatrix}$$

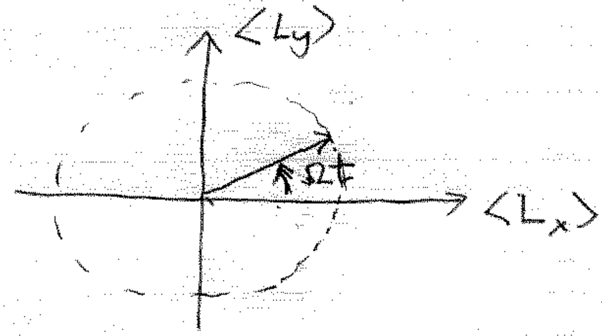
$$= \frac{1}{4\sqrt{2}}\left[e^{-i\Omega t}\ \ \sqrt{2}\ \ e^{i\Omega t}\right]\begin{bmatrix} \sqrt{2} \\ 2\cos\Omega t \\ \sqrt{2} \end{bmatrix} = \cos\Omega t$$

$$\langle\hat{l}_y\rangle_t = \langle\psi(t)|\hat{l}_y|\psi(t)\rangle = \frac{+i}{4\sqrt{2}}\left[e^{+i\Omega t}\ \ \sqrt{2}\ \ e^{i\Omega t}\right]\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} e^{-i\Omega t} \\ \sqrt{2} \\ e^{+i\Omega t} \end{bmatrix}$$

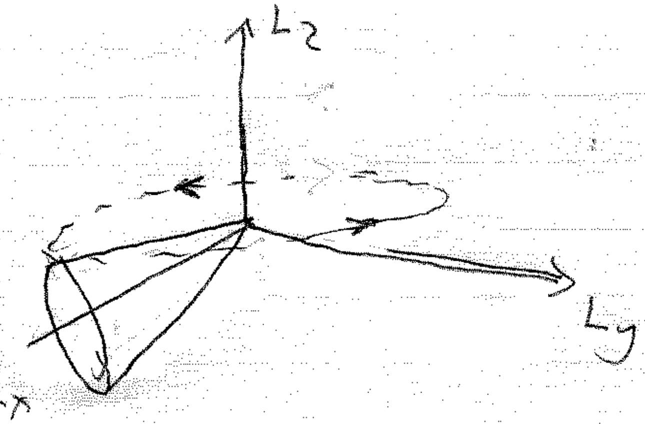
$$= \frac{+i}{4\sqrt{2}}\left[e^{-i\Omega t}\ \ \sqrt{2}\ \ e^{i\Omega t}\right]\begin{bmatrix} -\sqrt{2} \\ -i2\sin\Omega t \\ \sqrt{2} \end{bmatrix} = +\sin\Omega t$$

$$\langle\hat{l}_z\rangle_t = 0$$

Thus the mean value of $\langle \vec{L} \rangle$ rotates in the x-y plane at frequency $\Omega_L = \frac{\mu_B B}{\hbar}$



Of course the above picture can be somewhat misleading as one cannot specify exact values of all components of $\vec{L}$. Instead a more telling sketch is show below



At $t=0$ there is a definite projection of angular momentum along the x-axis and an uncertainty of the y-z components. As time proceeds, the direction of definite projection rotates about the -(z-axis) at frequency Ω_L.

(negative because $\vec{\mu} = -\mu_B \vec{l}$)

What about spin angular momentum?

Suppose we look at the magnetic moment associated with the electron's intrinsic spin

$$\hat{\vec{\mu}}_S = -g_s \mu_B \frac{\hat{\vec{S}}}{\hbar} = -\mu_B \hat{\vec{\sigma}}$$

Suppose at time $t=0$ $\quad |\psi(0)\rangle = |+\rangle = \dfrac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}}$

This is the state ~~with~~ with $\hat{\sigma}_x |+\rangle = +1\,|+\rangle$ (i.e. spin-up along x)

The Hamiltonian $\quad \hat{H} = -\hat{\vec{\mu}}_S \cdot \vec{B} = +\mu_B \underbrace{B_z}_{\text{(choose } \vec{B} \text{ in } z\text{-direction)}} \hat{\sigma}_z = \dfrac{\hbar\Omega}{2}\hat{\sigma}_z$

where $\Omega = g_s \mu_B B/\hbar$

$$\Rightarrow |\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle = e^{-i\frac{\Omega t}{2}\hat{\sigma}_z}\left(\frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}\left(e^{-i\Omega t/2}|\uparrow\rangle + e^{+i\Omega t/2}|\downarrow\rangle\right)$$

$$\doteq \frac{1}{\sqrt{2}}\begin{bmatrix} e^{-i\Omega t/2} \\ e^{i\Omega t/2} \end{bmatrix} \quad \text{(in standard basis)}$$

$$\langle\psi(t)|\hat{\sigma}_x|\psi(t)\rangle = \frac{1}{2}\begin{bmatrix} e^{+i\Omega t/2} & e^{i\Omega t/2} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} e^{-i\Omega t/2} \\ e^{i\Omega t/2} \end{bmatrix}$$

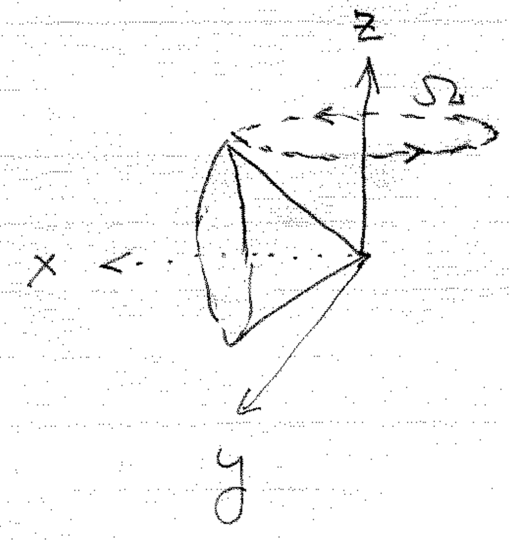
$$\Rightarrow \langle \hat{\sigma}_x \rangle_t = \frac{1}{2}\begin{bmatrix} e^{i\frac{\Omega t}{2}} & e^{-i\frac{\Omega t}{2}} \end{bmatrix} \begin{bmatrix} e^{i\Omega t/2} \\ e^{-i\Omega t/2} \end{bmatrix} = \cos\Omega t$$

$$\langle \hat{\sigma}_y \rangle_t = \frac{1}{2}\begin{bmatrix} e^{i\frac{\Omega t}{2}} & e^{-i\Omega t/2} \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} e^{-i\Omega t/2} \\ e^{+i\Omega t/2} \end{bmatrix}$$

$$= \frac{i}{2}\begin{bmatrix} e^{i\frac{\Omega t}{2}} & e^{-i\frac{\Omega t}{2}} \end{bmatrix} \begin{bmatrix} -e^{+i\Omega t/2} \\ e^{-i\Omega t/2} \end{bmatrix} = \sin\Omega t$$

$$\langle \hat{\sigma}_z \rangle_t = \frac{1}{2}\begin{bmatrix} e^{i\frac{\Omega t}{2}} & e^{-i\frac{\Omega t}{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} e^{-i\Omega t/2} \\ e^{i\Omega t/2} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} e^{i\Omega t/2} & e^{-i\Omega t/2} \end{bmatrix} \begin{bmatrix} e^{-i\Omega t/2} \\ -e^{+i\Omega t/2} \end{bmatrix} = 0$$



Same precession, only now at freq

$$\Omega = \frac{g_s \mu_B B}{\hbar} = 2\frac{\mu_B B}{\hbar}$$

(twice frequency as $l=1$ due to g-factor)

So for both orbital and spin angular momentum, the magnetic moment Larmor presesses about an applied constant $\vec{B}$ field, if the state is not an eigenstate of the $\vec{B}$ direction.

<u>But</u>, there is a subtle yet important quantum mechanical difference between orbital and spin angular momentum.

<u>Note</u>: the operator $e^{-i\Omega t \hat{J}_z/\hbar}$ produces a rotation of $\langle \hat{\vec{J}} \rangle$ about the z-axis by an angle $\theta = \Omega t$. More generally, the operator $e^{-i\theta(\vec{e}_n \cdot \hat{\vec{J}})/\hbar}$ rotates $\langle \hat{\vec{J}} \rangle$ by an angle θ about the axis $\vec{e}_n$ (unit vector)

Consider z-axis rotation and an arbitrary state

$$|\psi\rangle = \sum_{m=-j}^{j} c_m |j, m\rangle$$

After rotation, $|\tilde{\psi}\rangle = e^{-i\theta \hat{J}_z/\hbar} |\psi\rangle$

$$|\tilde{\psi}\rangle = \sum_{m=-j}^{j} e^{-im\theta} c_m |j, m\rangle$$

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After a rotation of 2π (i.e. $\theta = 2\pi$)

$$|\tilde{\psi}\rangle = \sum_{m=-j}^{j} e^{-im2\pi} c_m |j, m\rangle$$

Now if $m =$ integer $e^{-im2\pi} = 1$

$m =$ half-integer $e^{-im2\pi} = -1$

$\Rightarrow$ j integer $|\tilde{\psi}\rangle = |\psi\rangle$ } for $\theta = 2\pi$
j half-integer $|\tilde{\psi}\rangle = -|\psi\rangle$ }

Thus, for <u>half-integer spin</u> $|\psi\rangle \Rightarrow -|\psi\rangle$ after a 2π rotation, (i.e. the state picks up a π-phase shift $e^{i\pi} = -1$). Of course the overall phase of the state is irrelevant, so the physical expectation value are the same. However, this phase can have physical relevance as we will see.

The -1 for ~~spin~~ ½-integer vs. whole integer upon 2π rotation is one of the defining features of quantum mechanics.