Physic 492 Quantum II

Lecture 15: Entangled States, EPR, and Bell

- Measurement: We have studied the fundamental axiom of quantum theory – the von Neumann projection postulate. Given an observable $\hat{O}$ with eigenspectrum $\hat{O}|o\rangle = o|o\rangle$, if we measure and find eigenvalue o for state $|\psi\rangle$, then the post-measure state is projected onto $|o\rangle$

$$\underset{\text{before}}{|\psi\rangle} \longrightarrow \underset{\text{after}}{|\tilde{\psi}\rangle} = \underset{\substack{\uparrow \\ \text{projection}}}{\hat{\mathbb{P}}_o}|\psi\rangle \quad \text{(unnormalized)}$$

The projection operator $\hat{\mathbb{P}}_o = |o\rangle\langle o|$

So $|\tilde{\psi}\rangle = |o\rangle \underbrace{\langle o|\psi\rangle}_{\text{pro amplitude}}$

pro amplitude: $\quad p_o = |\langle o|\psi\rangle|^2$
$\qquad\qquad\qquad\qquad = \langle\psi|\hat{\mathbb{P}}_o|\psi\rangle$

Normalized $|\tilde{\psi}\rangle = |o\rangle$

- <u>Measurement on a composite quantum system</u>

Consider a composite quantum system with two parts ("bipartite" system)

$$\mathcal{H}_{AB} = h_A \otimes h_B$$

In principle, one can measure one or the other parts, or both.

Consider, for example, two spin-½ particles in the state

$$|\Psi_{AB}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_A \otimes |\downarrow\rangle_B - |\downarrow\rangle_A \otimes |\uparrow\rangle_B \right)$$

This is an <u>entangled state</u>

Suppose we measure $\hat{\sigma}_z^{(A)}$ (i.e. the z-component of spin for particle A). There are two eigenstates $|\uparrow\rangle_A$ and $|\downarrow\rangle_A$.

If we find:

- $|\uparrow_A\rangle$ then post-measurement the unnormalized state is

$$|\tilde{\Psi}_{AB}\rangle = \hat{\mathbb{P}}^{(A)}_{\uparrow}|\Psi_{AB}\rangle = |\uparrow_A\rangle\langle\uparrow_A|\Psi_{AB}\rangle$$

$$= \frac{1}{\sqrt{2}}\left(|\uparrow_A\rangle\underbrace{\langle\uparrow_A|\uparrow_A\rangle}_{1} \otimes |\downarrow_B\rangle - |\uparrow_A\rangle\underbrace{\langle\uparrow_A|\downarrow_A\rangle}_{0} \otimes |\uparrow_B\rangle \right)$$

- Unnormalized post-measurement state for $|\uparrow_A\rangle$

$$|\tilde{\Psi}_{AB}\rangle = \frac{1}{\sqrt{2}} |\uparrow_A\rangle \otimes |\downarrow_B\rangle$$

Normalized $\quad |\tilde{\Psi}_{AB}\rangle = |\uparrow_A\rangle \otimes |\downarrow_B\rangle$

- If we find $|\downarrow_A\rangle$ then post-measurement

$$|\tilde{\Psi}_{AB}\rangle = \hat{P}^{(A)}_{\downarrow} |\Psi_{AB}\rangle = |\downarrow_A\rangle\langle\downarrow_A|\Psi_{AB}\rangle$$

$$= \frac{1}{\sqrt{2}} \Big(|\downarrow_A\rangle\langle\downarrow_A|\uparrow_A\rangle \otimes |\downarrow_B\rangle -$$

$$|\downarrow_A\rangle\langle\downarrow_A|\downarrow_A\rangle \otimes |\uparrow_B\rangle \Big)$$

$$= -\frac{1}{\sqrt{2}} \; |\downarrow_A\rangle \otimes |\uparrow_B\rangle$$

Normalized

$$|\tilde{\Psi}_{AB}\rangle = |\downarrow_A\rangle \otimes |\uparrow_B\rangle$$

Thus, in summary, given the entangled state

$$|\Psi_{AB}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_A \downarrow_B\rangle - |\downarrow_A \uparrow_B\rangle\right)$$

if we measure <u>only</u> spin-A, then

- Find $|\uparrow_A\rangle \Rightarrow$ post-measurement $|\Psi_{AB}\rangle = |\uparrow_A \downarrow_B\rangle$
- Find $|\downarrow_A\rangle \Rightarrow$ post-measurement $|\Psi_{AB}\rangle = |\downarrow_A \uparrow_B\rangle$

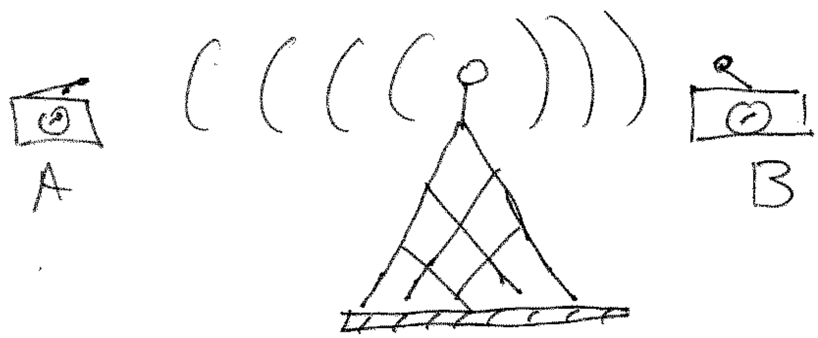
<u>Note:</u> There is a strong <u>correlation</u> between the value of the spin for A and B. If we find $|\uparrow_A\rangle$ then the post-measurement state is $|\uparrow_A\rangle \otimes |\downarrow_B\rangle$. Thus $\hat{\sigma}_z^{(B)}$ has a <u>definite out come</u> $|\downarrow_B\rangle$. Similarly, if we measure A and find $|\downarrow_A\rangle$, then B will find $|\uparrow_B\rangle$. The values of $\hat{\sigma}_z^{(A)}$ and $\hat{\sigma}_z^{(B)}$ are <u>tied</u> together. <u>They are perfectly (anti) corrlated</u>

$$|\uparrow_A\rangle \Longleftrightarrow |\downarrow_B\rangle$$

$$|\downarrow_A\rangle \Longleftrightarrow |\uparrow_B\rangle$$

Correlated outcomes are not unfamiliar in classical physics. Such outcomes are causally related; the value of A and B are tied together through cause and effect.

Two radios, tuned to the same station, will receive the same signal because the waves they receive were generated from the same ~~antenna~~ source



The signal at A and B are <u>correlated</u>. A can predict the signal B receives if he tunes to the same station as A because A knows how the signal was prepared.

• EPR

The correlations associated with entangled quantum states are in fact very different from the classical correlations described above. This point was first brought to light by Einstein - Podolsky - and Rosen (today known as EPR) in a famous paper

"Can Quantum - Mechanical Descriptions of Physical Reality be Considered Complete?",

Physical Review vol 47, pages 777-780 (1935).

Einstein is well known for his distate for the "new" quantum theory. His most famous quote on the subject is "God does not roll dice with the universe!". Einstein was not, in fact, unhappy with the stochastic nature of quantum measurement. He was a master of classical statistical physics. His problem was with the almost "mystical" view of Q.M. establish in the Copenhaggen school, triumphed by Bohr.

Einstein believed, as is natural, that the randomness observed in quantum mechanics was due to incomplete information. That is there are some "hidden variables", whose values, if known, would determine the outcome of measurements. Because there values are "hidden", our ignorance becomes a lack of predictive power, and thus we resort to probabilities to quantify this ignorance. The EPR paradox puts a very fine point on the question of hidden variables. It ties together notions of causality with measurement and hidden variables.

The idea they wanted to get at is "objective properties". that is, measurements are uncovering properties of physical system that "exist", independent of the measurement.

These objective properties they called "elements of physical reality" which they defined as:

"If without, in any way, disturbing a system, we can predict, with certainty (i.e. with unit probability), the value of a physical quantity, then there exists an element of physical reality corresponding to this quantity"

The argument, put forward by EPR (described below) shows that there definition of element of reality is inconsistent with quantum theory, and thus quantum theory must not be the "complete theory" of nature.

(Next Page)

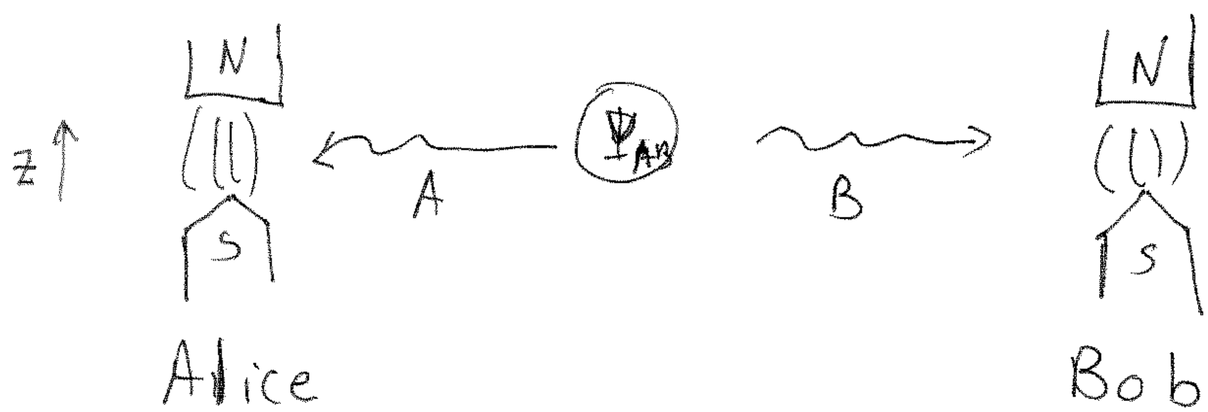
EPR - Bohm

We consider a version of the EPR paradox put forward by David Bohm (Phy Rev 108 1070 1957) one of the great "quantum philosophers".

Consider the entangled state of two spins

$$|\Psi_{AB}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_A \otimes |\downarrow\rangle_B - |\downarrow\rangle_A \otimes |\uparrow\rangle_B\right)$$

Suppose the two spin fly off to distant parts of the universe and then measured



Now because of the perfect (anti) correlation Alice can predict the $\hat{\sigma}_z^{(B)}$ of Bob.

Because they are at distant positions Alice can predict the value Bob will measure without any possibility of a signal going from Alice to Bob.

~~The value~~ Alice's measuring process cannot effect the value Bob measures.

This is the assumption of "Einstein causality".

Two events are causally connected only if a signal traveling no faster than the speed of light ~~can~~ can communicate between them.

Thus according to EPR $\hat{\sigma}_z^B$ is an "element of reality". The z-component of spin for Bob is objectly real.

Bob merely "discovers" what that value is when he measures it.

Now it turns out that (see homework)

$$|\Psi_{AB}\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow_n\rangle_A \otimes |\downarrow_n\rangle_B$$

$$- |\downarrow_n\rangle_A \otimes |\uparrow_n\rangle_B\Big)$$

where $|\uparrow_n\rangle = \cos\frac{\theta}{2}\,|\uparrow_z\rangle + e^{i\phi}\sin\frac{\theta}{2}\,|\downarrow_z\rangle$

$|\downarrow_n\rangle = \sin\frac{\theta}{2}\,|\uparrow_z\rangle - e^{i\phi}\cos\frac{\theta}{2}\,|\downarrow_z\rangle$

That is, no matter what direction $\vec{e}_n$ the spin is measured the values are perfectly anticorrelated

Thus, for example,

$$|\Psi_{AB}\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow_x\rangle_A \otimes |\downarrow_x\rangle_B - |\downarrow_x\rangle_A \otimes |\uparrow_x\rangle_B\Big)$$

So if Alice measures $\hat{\sigma}_x^{(A)}$ she knows <u>instaneously</u> the value Bob will find.

$\Rightarrow$ $\hat{\sigma}_x^{(B)}$ is objectively real according to EPR.

Thus, according to EPR, both $\hat{\sigma}_z^{(B)}$ and $\hat{\sigma}_x^{(B)}$ are objectively real. Their values exist, objectively in the universe, and our measurement discovers what they are.

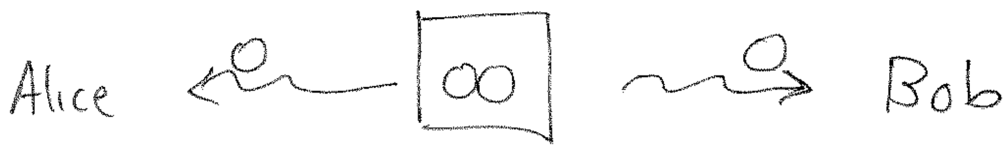
But, according to quantum theory $\hat{\sigma}_z^{(B)}$ and $\hat{\sigma}_x^{(B)}$ <u>don't commute</u>.

Thus, in quantum theory, it is <u>impossible</u> to write a state which simultaneously assigns definite values to $\hat{\sigma}_x^{(B)}$ and $\hat{\sigma}_z^{(B)}$. Thus, according to EPR, quantum mechanics is <u>incomplete</u>

Hence the paradox.

An example of classical correlations and hidden variables

Suppose someone puts two marbles in a box, one red the other green. This preparer secretly sends one marble to Alice the other to Bob, located at opposite ends of the room



Now, if Alice finds a red ball she knows instantly that Bob find a green and vice versa. No surprise here. the marble had a definite objective color on its way to Bob. Alice could predict its value once she saw her marble because she knew how they were correlated. A priori she did not know which one she would receive because of the hidden variable (the preparer's choice of which ball to send).

John Bell:

The EPR result was debated on philosophical grounds but not resolved in any way. Thirty years later John Bell reëxamined the EPR paradox and the question of hidden variables. Bell took the EPR arguement to its logical conclusion and deduce a <u>quantitive test</u>. He moved philosophy to the laboratory.

Consider the Bohm version of EPR. Alice and Bob make measurements of $\hat{\sigma}_n$ which has eigenvalue ± 1. A "local hidden variable" assigns an objective value $+1$ or -1 to this observable, which depends on the axis $\vec{e}_n$ and the hidden variables, which we denote collectively as λ

Thus,

Alice finds a value $A(\vec{e}_a, \lambda)$

Bob finds a value $B(\vec{e}_a, \lambda)$

Note: the value A <u>does not</u> depend on $\vec{e}_b$

the value B <u>does not</u> depend on $\vec{e}_a$

this is a statement of <u>locality</u>.

Bell considered average values which arise from our uncertain knowledge of the hidden variables λ. We assign a probability distribution to the hidden variables: $P(\lambda)$.

$$\Rightarrow \quad \underset{\substack{\text{Hidden}\\\text{variable}}}{C(a,b)} = \langle \sigma_a \sigma_b \rangle_{\substack{\text{Hidden}\\\text{variables}}}$$

$$= \int d\lambda \, P(\lambda) \, A(\vec{e}_a, \lambda) \, B(\vec{e}_b, \lambda)$$

The local hidden variable model is supposed to reproduce the quantum mechanical expectation value

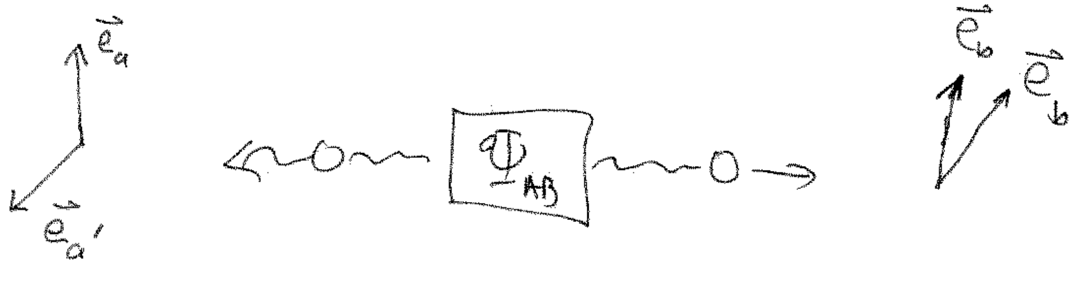
$$C_{\text{Quantum}}(a,b) = \langle \hat{\sigma}_a \hat{\sigma}_b \rangle_{\text{Quantum}} = \langle \Psi_{AB} | \hat{\sigma}_a^{(A)} \otimes \hat{\sigma}_b^{(B)} | \Psi_{AB} \rangle$$

For $|\Psi_{AB}\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle_A \otimes |\downarrow\rangle_B - |\downarrow\rangle_A \otimes |\uparrow\rangle_B \right)$

$$\langle \hat{\sigma}_a \hat{\sigma}_b \rangle_{\text{Quantum}} = - \vec{e}_a \cdot \vec{e}_b \quad \text{(see Homework)}$$

<u>CHSH:</u> The form of Bell's inequality we consider here was due to Clauser-Holt-Shimony-Horne (CHSH). It is a generalization of the one originally derived by Bell, but more amenable to experiments.

Consider a scenario in which Alice measures her spin along <u>either</u> $\vec{e}_a$ or $\vec{e}_{a'}$, Bob along $\vec{e}_b$ or $\vec{e}_{b'}$



Define the observable:

$$\hat{S} = \hat{\sigma}_a \otimes (\hat{\sigma}_{b'} - \hat{\sigma}_b) + \hat{\sigma}_{a'} \otimes (\hat{\sigma}_{b'} + \hat{\sigma}_b)$$

$$\Rightarrow \quad = C_a(a, b') - C(a,b) + C(a', b') + C(a', b)$$

Consider the values of S in a local hidden variable model

$$S_\lambda = A(\vec{e}_a, \lambda) \left[B(\vec{e}_{b'}, \lambda) - B(\vec{e}_b, \lambda) \right] + A(\vec{e}_{a'}, \lambda) \left[B(\vec{e}_{b'}, \lambda) + B(\vec{e}_b, \lambda) \right]$$

In this model, all values are "objectively real" and taken on value $+1$ or -1 with some probability determined by λ. If $B(\vec{e}_{b'}, \lambda) = B(\vec{e}_b, \lambda)$ the first term vanishes and $S_\lambda = 2A(\vec{e}_{a'}, \lambda) B(\vec{e}_b, \lambda)$.

If $B(\vec{e}_{b'}, \lambda) = -B(\vec{e}_b, \lambda)$, $S_\lambda = 2A(\vec{e}_a, \lambda) B(\vec{e}_{b'}, \lambda)$

Thus, in any run of the experement $S_\lambda = \pm 2$

$$\Rightarrow \langle S_\lambda \rangle_{\substack{\text{Hidden} \\ \text{variable}}} = \int d\lambda \, P(\lambda) \, S_\lambda \quad \text{is } \underline{\text{bounded}}$$

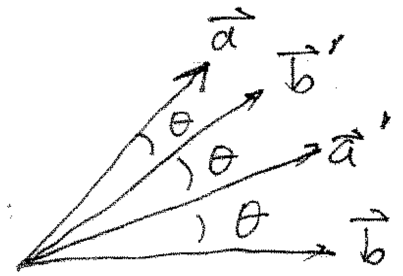
$$\Rightarrow \boxed{-2 \le \langle S_\lambda \rangle_{\substack{\text{Local} \\ \text{Hidden}}} \le +2}$$

This is the CHSH Bell inequality

The prediction of quantum physics:

$$\langle \hat{S} \rangle_{QM} = -\left(\vec{e}_a \cdot \vec{e}_b{}' - \vec{e}_a \cdot \vec{e}_b + \vec{e}_{a'} \cdot \vec{e}_{b'} + \vec{e}_{a'} \cdot \vec{e}_{b}\right)$$

Consider a set of possible measurement directions:

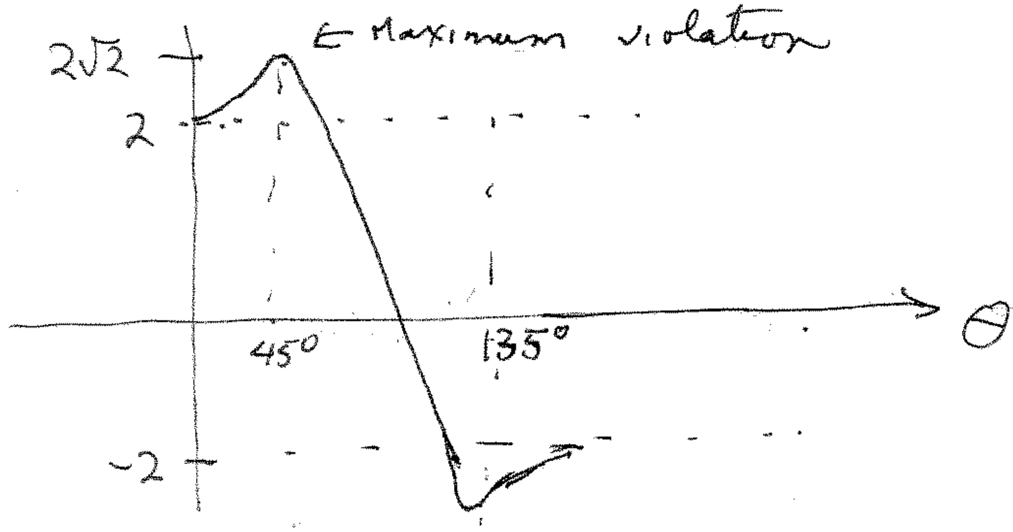


$$\Rightarrow \quad \langle \hat{S} \rangle_{Q.M.} = 3\cos\theta - 3\cos 3\theta$$

For small θ $\quad \langle \hat{S} \rangle_{QM} \approx 3\left(1 - \frac{\theta^2}{2}\right) - \left(1 - \frac{9}{2}\theta^2\right)$

$$= 2 + 3\theta^2$$

$$\Rightarrow \boxed{\langle \hat{S} \rangle_{Q.M.} > 2 \quad \Rightarrow \quad \text{Quantum Mechanics Violates Bell inequality}}$$



Quantum entanglement <u>cannot</u> be a local hidden variable model.

The upshot of this is that, contrary to EPR's assumption, a L.H.V.M. cannot reproduce all of quantum theory. If we want to devise a hidden variable model, it must be non local, as in the Bohmian theory. Alternatively, we must content ourselves with the fact that the microscopic properties are <u>not</u> objectly <u>real</u>. We find a random result correlated with the measurement apparatus as Bohr described. Only the <u>whole system</u>, micro plus macro apparatus is physically objective.

<u>Experiments:</u>
There have been numerous experiments that have shown violations of a Bell inequality in agreement with Q.M. Most have been done with photons, measuring the polarization (a two state system which maps onto spin 1/2). Notable examples are:

- Kwiat et al PRL <u>60</u> 773 (1992)
- Freedman + Clauser: PRL <u>28</u> 938 (1972) (The first try)
- Aspect et al: PRL <u>47</u> 460 (1981) (The first observed violation of Bell)