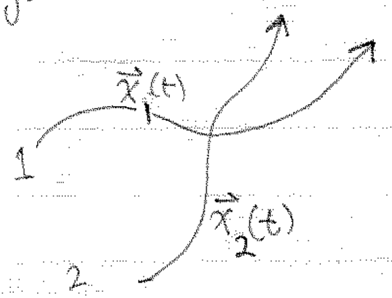
Physic 492 - Quantum II

Lecture 18: Identical particles, spin, statistics

Background: In classical physics we can follow identifiable trajectories of each particle



In quantum physics particles no longer follow well defined trajectories but instead are defined by wave functions. If the wave packet of particle 1 overlaps that of particle 2, and the particles are identical (e.g. two electrons), then there is in principle no information

available to ~~the~~ distinguish which particle is at which position. Thus, the joint probability density must be invariant w.r.t. to exchange of the two particles

$$P(\vec{x}_1, \vec{x}_2) = P(\vec{x}_2, \vec{x}_1)$$

particle 1 @ $\vec{x}_1$ particle 2 @ $\vec{x}_2$ particle 1 @ $\vec{x}_2$ particle 2 @ $\vec{x}_1$

> Exchange symmetry of identical particles is a fundamental axiom of quantum theory

Let us ignore spin for the moment (we'll come back to this in a second).

Consider a product wave function of particle 1 and 2

$$\Psi(x_1, x_2) = \underbrace{\phi_A(x_1)}_{\substack{\text{particle 1} \\ \text{in "orbital" } \phi_A}} \underbrace{\phi_B(x_2)}_{\substack{\text{particle 2 in} \\ \text{"orbital" } \phi_B}}$$

This wave function does not lead to a joint prob. distribution invariant under exchange

$$P(x_1, x_2) = |\Psi(x_1, x_2)|^2 = |\phi_A(x_1)|^2 |\phi_B(x_2)|^2$$

$$\neq P(x_2, x_1) = |\phi_A(x_2)|^2 |\phi_B(x_1)|^2$$

We must have a wave function for two identical particles that is an eigenstate of the particles exchange operator

action $\hat{\mathbb{P}}_{12} \Psi(x_1, x_2) = \Psi(x_2, x_1)$

Note $\hat{\mathbb{P}}_{12} \hat{\mathbb{P}}_{12} \Psi(x_1, x_2) = \Psi(x_1, x_2)$

$\Rightarrow (\hat{\mathbb{P}}_{12})^2$ has eigenvalue $+1$ $\forall$ states

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Thus, $\hat{\mathbb{P}}_{12}$ has eigenvalues $\underline{\pm 1}$

if $\hat{\mathbb{P}}_{12}\,\Psi(x_1,x_2) = +1\;\Psi(x_1,x_2)$: Symmetric under exchange $\Rightarrow$ Boson

if $\hat{\mathbb{P}}_{12}\,\Psi(x_1,x_2) = -1\;\Psi(x_1,x_2)$: Antisymmetric under exchange $\Rightarrow$ Fermion

Thus we must symmetrize (antisymmetrize) the wave function when dealing with systems of identical bosons (fermions).

Given one particle in "orbital" ϕ_A and the other in "ϕ_B" the proper symmetrized/anti-sym joint wave function is

boson

$$\Psi(x_1,x_2) = \frac{1}{\sqrt{2}}\left(\phi_A(x_1)\,\phi_B(x_2) \pm \phi_B(x_1)\,\phi_A(x_2)\right)$$

fermion

The wave function is a ~~sg~~ superposition of the two possibilities

In Dirac notation

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\phi_A\rangle \otimes |\phi_B\rangle \pm |\phi_B\rangle \otimes |\phi_A\rangle\right)$$

Aside: Are we now bound only to write joint wavefunctions in the universe? Must we, for example, consider electrons on the moon when measuring electrons on earth? Of course not!

Consider the joint probability density for the wave function

$$|\psi(x_1, x_2)|^2 = \left|\frac{1}{\sqrt{2}}\left(\phi_A(x_1)\,\phi_B(x_2) \pm \phi_B(x_1)\,\phi_A(x_2)\right)\right|^2$$

$$= \frac{1}{2}\left(|\phi_A(x_1)|^2 |\phi_B(x_2)|^2 + |\phi_B(x_1)|^2 |\phi_A(x_2)|^2 \pm \underbrace{\left(\phi_A(x_1)\,\phi_B^*(x_1)\,\phi_A^*(x_2)\,\phi_B(x_2) + c.c\right)}_{\text{exchange density}} \right)$$

Consider the expection value of an operator

$\hat{A}(x_1, x_2) = \hat{A}(x_2, x_1)$ (treats particles identically)

$$\Rightarrow \langle \hat{A} \rangle = \int dx_1 dx_2 \, A(x_1, x_2) \, |\psi(x_1, x_2)|^2$$

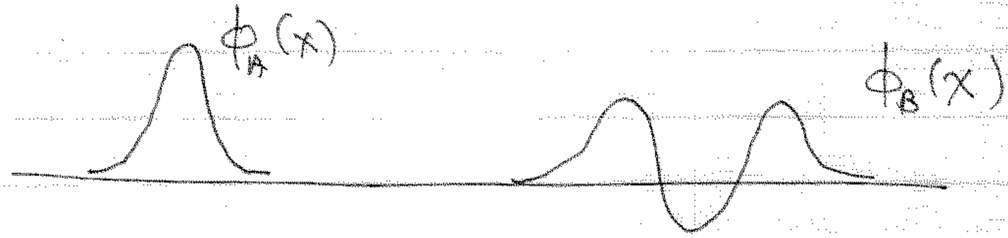
$$= \int dx_1 dx_2 \, A(x_1, x_2) \, |\phi_A(x_1)|^2 |\phi_B(x_2)|^2$$

$$\pm \left(\int dx_2 dx_2 \, A(x_1, x_2) \, \phi_A(x_1)\,\phi_B^*(x_1)\,\phi_A^*(x_2)\,\phi_B(x_2) + c.c. \right)$$

The first term is known as the "direct term" and the second line the "exchange term".

Now suppose $\phi_A(x)$ and $\phi_B(x)$ have <u>negligible</u> <u>overlap</u>

e.g.



$$\Rightarrow \phi_A(x_1)\,\phi_B^*(x_1) \approx 0 \qquad \phi_A^*(x_2)\,\phi_B(x_2) \approx 0$$

$\Longrightarrow$ <u>Exchange term negligible</u>

$$\Rightarrow \langle \hat{A} \rangle \approx \int dx_1\, dx_2\; A(x_1, x_2)\; |\phi_A(x_1)|^2\, |\phi_B(x_2)|^2$$

Can ignore symmetrization

Thus, ~~if~~ the wave functions for the two particles don't overlap they are <u>effectively</u> ~~[illegible]~~ <u>distinguishable</u> by their position and we can do quantum mechanics without worrying about the nature of identical particles.

On the other hand, in atoms molecules, solid with mulitple electrons we <u>must</u> take into account the nature of identical particles.

Spin and identical particles

Now we return to the question of spin, a fundamental ingredient in the theory of identical particles.

> **Fundamental theorem of spin and statistics:**
> particles with integer spin = bosons
> Particles with ½-integer spin = fermions

The connection between spin and statistics follows from relativistic quantum theory. Here we take it as a fact.

Aside: Elementary particles come in both integer and half integer varieties

Matter: Leptons (e.g electrons), Quarks (e.g nuclear matter), Neutrinos
$\rightarrow$ Spin ½ = fermions

Force fields: e.g. photons = spin 1 = Bosons
gluons
Z-boson

Composite particles

Even # fermions = Boson

Odd # fermions = Fermion

Pauli exclusion principle

Stated loosely, no two electrons can be in the same quantum state.

This is a consequence of the fact that electrons are fermions and thus the total state must be antisymmetric under exchange.

$$\frac{1}{\sqrt{2}}\left(|\phi_A\rangle_1 \otimes |\phi_B\rangle_2 - |\phi_B\rangle_1 \otimes |\phi_A\rangle_2\right) \rightarrow 0$$

when $|\phi_A\rangle = |\phi_B\rangle$

Now when writing the ~~total~~ total state we must include <u>both</u> motional and spin degrees of freedom.

Suppose our state is a product of ~~sp~~ motion and spin

$$\Psi(x_1, x_2) = \phi_{motion}(x_1, x_2) \otimes |\chi_{12}\rangle_{spin}$$

Require $\hat{P}_{12}\Psi(x_1,x_2) = \hat{P}_{12}^{motion}\,\phi_{motion}(x_1,x_2) \otimes \hat{P}_{12}^{spin}\,|\chi_{12}\rangle_{spin}$

$$= \overset{\text{boson}}{\underset{\text{fermion}}{\pm}}\ \Psi(x_1, x_2)$$

	ϕ_{motion}	χ_{spin}
Bose	sym	sym
	anti-sym	anti-sym
Fermi	sym	anti-sym
	anti-sym	sym

For bosons, the spin and motional state must have the same symmetry. For fermions they must be opposite.

Example, two spin-1/2 particles

Suppose (Motional state) $\otimes$ (Spin States)

$$\underbrace{\underbrace{\frac{1}{\sqrt{2}}\left(\phi_A(x_1)\phi_B(x_2) + \phi_B(x_1)\phi_A(x_2)\right)}_{\text{symmetric motion}} \otimes \underbrace{\frac{1}{\sqrt{2}}\left(|\uparrow_1\downarrow_2\rangle - |\downarrow_1\uparrow_2\rangle\right)}_{\text{antisymmetric spin}}}_{\text{overall anti-symmetric}}$$

Note: Here the spins are coupled into a spin singlet $S=0$

$$\frac{1}{\sqrt{2}}\left(\phi_A(x_1)\phi_B(x_2) - \phi_B(x_1)\phi_B(x_2)\right) \otimes \begin{cases} |\uparrow_1\uparrow_2\rangle \\ \frac{1}{\sqrt{2}}\left(|\uparrow_1\downarrow_2\rangle + |\downarrow_1\uparrow_2\rangle\right) \\ |\downarrow_1\downarrow_2\rangle \end{cases}$$

anti-symmetric motion

symmetric spin

Note: Here the three possible spin states are the spin triplet, $S=1$

Note: Two spins in a singlet state have a positive exchange density $\Rightarrow$ enhanced probability to be at the same position

Two spins in a triplet state have a negative exchange density $\Rightarrow$ decreased probability to be at the same position.