Lecture 18: Identical particles, spin, statistics

Background: In classical physics we can follow identifiable trajectories of each particle.

In quantum physics particles no longer follow well-defined trajectories but instead are defined by wave functions. If the wave packet of particle 1 overlaps that of particle 2, and the particles are identical (e.g. two electrons), then there is in principle no information available to distinguish which particle is at which position. Thus, the joint probability density must be invariant w.r.t. to exchange of the two particles.

\[ P(\vec{x}_1, \vec{x}_2) = P(\vec{x}_2, \vec{x}_1) \]

Exchange symmetry of identical particles is a fundamental axiom of quantum theory.
Let us ignore spin for the moment (we'll come back to this in a second).

Consider a product wave function of particle 1 and 2

\[ \psi(x_1, x_2) = \phi_A(x_1) \phi_B(x_2) \]

This wave function does not lead to a joint probability distribution invariant under exchange.

\[ P(x_1, x_2) = |\psi(x_1, x_2)|^2 = |\phi_A(x_1)|^2 |\phi_B(x_2)|^2 \]

\[ \neq P(x_2, x_1) = |\phi_A(x_2)|^2 |\phi_B(x_1)|^2 \]

We must have a wave function for two identical particles that is an eigenvector of the particles exchange operator.

Action \[ \hat{P}_{12} \psi(x_1, x_2) = \psi(x_2, x_1) \]

Note \[ \hat{P}_{12} \hat{P}_{12} \psi(x_1, x_2) = \psi(x_1, x_2) \]

\[ \Rightarrow (\hat{P}_{12})^2 \text{ has eigenvalue } +1 \text{ for states} \]

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Thus, \( \hat{P}_{12} \) has eigenvalues \( \pm 1 \)

- If \( \hat{P}_{12} \Psi(x_1, x_2) = +1 \) \( \Psi(x_1, x_2) \): Symmetric under exchange \( \Rightarrow \) boson
- If \( \hat{P}_{12} \Psi(x_1, x_2) = -1 \) \( \Psi(x_1, x_2) \): Antisymmetric under exchange \( \Rightarrow \) fermion

Thus, we must symmetrize (antisymmetrize) the wave function when dealing with systems of identical bosons (fermions).

Given one particle in "orbital" \( \phi_A \) and the other in "\( \phi_B \)" the proper symmetrized/antisym joint wave function is

\[
\Psi(x_1, x_2) = \frac{1}{\sqrt{2}} \left( \phi_A(x_1) \phi_B(x_2) + \phi_B(x_1) \phi_A(x_2) \right)
\]

The wave function is a superposition of the two possibilities

In Dirac notation

\[
|\Psi\rangle = \frac{1}{\sqrt{2}} (|\phi_A\rangle \otimes |\phi_B\rangle \pm |\phi_B\rangle \otimes |\phi_A\rangle)
\]
Aside: Are we now bound only to write joint wave functions in the universe? Must we, for example, consider electrons on the moon when measuring electrons on earth? Of course not!

Consider the joint probability density for the wave function

\[ |\psi(x_1, x_2)|^2 = \frac{1}{2} \left( |\phi_A(x_1)|^2 |\phi_B(x_2)|^2 + |\phi_B(x_1)|^2 |\phi_A(x_2)|^2 \right) \]

\[ + \left( \phi_A(x_1) \phi_B(x_2) \phi_A^*(x_2) \phi_B^*(x_2) + \text{c.c.} \right) \]

exchange density

Consider the expectation value of an operator \( A(x_1, x_2) = A(x_2, x_1) \) (treats particles identically)

\[ \langle A \rangle = \int dx_1 dx_2 A(x_1, x_2) |\psi(x_1, x_2)|^2 \]

\[ = \int dx_1 dx_2 A(x_1, x_2) |\phi_A(x_1)|^2 |\phi_B(x_2)|^2 \]

\[ \pm \left( \int dx_2 dx_1 A(x_1, x_2) \phi_A(x_1) \phi_B(x_2) \phi_A^*(x_2) \phi_B^*(x_2) + \text{c.c.} \right) \]
The first term is known as the "direct term" and the second line the "exchange term".

Now suppose $\phi_A(x)$ and $\phi_B(x)$ have negligible overlap.

$$\Rightarrow \phi_A(x_1) \phi_B^*(x_1) \approx 0 \quad \phi_A^*(x_2) \phi_B(x_2) \approx 0$$

$\Rightarrow$ Exchange term negligible.

$$\Rightarrow \langle \hat{A} \rangle \approx \int dx_1 \, dx_2 \, A(x_1, x_2) \, |\phi_A(x_1)|^2 \, |\phi_B(x_2)|^2$$

Can ignore symmetrisation.

Thus, if the wave functions for the two particles don't overlap they are effectively distinguishable by their position and we can do quantum mechanics without worrying about the nature of identical particles.

On the other hand, in atoms, molecules, solids with multiple electrons we must take into account the nature of identical particles.
Spin and identical particles

Now we return to the question of spin, a fundamental ingredient in the theory of identical particles.

**Fundamental theorem of spin and statistics:**
- Particles with integer spin = bosons
- Particles with half-integer spin = fermions

The connection between spin and statistics follows from relativistic quantum theory. Here we take it as a fact.

Aside: Elementary particles come in both integer and half-integer varieties.

**Matter:** Leptons (e.g. electrons), Quarks (e.g. nuclear matter), Neutrinos

\[ \implies \text{Spin} \frac{1}{2} = \text{fermions} \]

**Force fields:** e.g. photon = spin 1 = Bosons
- Gluons
- Z-boson

**Composite particles**
- Even # of fermions = Boson
- Odd # of fermions = Fermions
Pauli exclusion principle

Stated loosely, no two electrons can be in the same quantum state.

This is a consequence of the fact that electrons are fermions and thus the total state must be antisymmetric under exchange.

\[ \frac{1}{\sqrt{2}} (|\phi_1^A\rangle \otimes |\phi_2^B\rangle - |\phi_1^B\rangle \otimes |\phi_2^A\rangle) \rightarrow \text{O} \]

when \( |\phi_A^A\rangle = |\phi_B^B\rangle \)

Now when writing the total state we must include both motional and spin degrees of freedom.

Suppose our state is a product of motional and spin.

\[ \Phi(x_1, x_2) = \phi_{\text{motion}}(x_1, x_2) \otimes |\chi_{12}\rangle_{\text{spin}} \]

Require: \( \hat{P}_{12} \Phi(x_1, x_2) = \hat{P}_{12} \phi_{\text{motion}}(x_1, x_2) \otimes \hat{P}_{12} |\chi_{12}\rangle_{\text{spin}} \)

For bosons, the boson is \(+\), fermion is \(-\). They must have the same symmetry. For fermions, they must be opposite.

<table>
<thead>
<tr>
<th></th>
<th>(\phi_{\text{motion}})</th>
<th>(\chi_{\text{spin}})</th>
<th>For bosons, the spin and motional state must have the same symmetry. For fermions, they must be opposite.</th>
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<tr>
<td>Boson</td>
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<td>Fermion</td>
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<td></td>
<td>anti-sym</td>
<td>sym</td>
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Example: two spin-\(\frac{1}{2}\) particles

Suppose

\[
\left\{ \frac{1}{\sqrt{2}} \left( \phi_A(x_1) \phi_B(x_2) + \phi_B(x_1) \phi_A(x_2) \right) \right\} \otimes \left\{ \frac{1}{\sqrt{2}} \left( |\uparrow, \downarrow_2\rangle - |\downarrow, \uparrow_2\rangle \right) \right\}
\]

symmetric motion
antisymmetric spin

overall anti-symmetric

Note: Here the spins are coupled into a spin singlet, \(S=0\).

\[
\left\{ \frac{1}{\sqrt{2}} \left( \phi_A(x_1) \phi_B(x_2) - \phi_B(x_1) \phi_A(x_2) \right) \right\} \otimes \left\{ \frac{1}{\sqrt{2}} \left( |\uparrow, \downarrow_2\rangle + |\downarrow, \uparrow_2\rangle \right) \right\}
\]

anti-symmetric motion

\[
\frac{1}{\sqrt{2}} \left( |\uparrow, \downarrow_2\rangle + \sqrt{3} |\downarrow, \uparrow_2\rangle \right)
\]
symmetric spin

Note: Here the three possible spin states are the spin triplet, \(S=1\).

Note: Two spins in a singlet state have a positive exchange density \(
\Rightarrow \) enhanced probability to be at the same position.

Two spins in a triplet state have a negative exchange density \(
\Rightarrow \) decreased probability to be at the same position.