Problem 1: Unitary operators (10 Points)

An important class of operators are unitary, defined as those that preserve inner product, i.e. if $\langle \tilde{\psi} | \tilde{\phi} \rangle = \langle \psi | \phi \rangle$ and $\langle \tilde{\phi} | \tilde{\psi} \rangle = \langle \phi | \psi \rangle$, then $\langle \tilde{\phi} | \tilde{\psi} \rangle = \langle \phi | \psi \rangle$ and $\langle \tilde{\psi} | \tilde{\phi} \rangle = \langle \psi | \phi \rangle$.

(a) Show that unitary operators $\hat{U}^* \hat{U} = \hat{U} \hat{U}^* = \hat{1}$ (i.e. the adjoint is the inverse).

(b) Consider $\hat{U} = \exp(i\hat{A})$, where $\hat{A}$ is a Hermitian operator. Show that $\hat{U}^* = \exp(-i\hat{A})$ and thus show $\hat{U}$ is unitary.

(c) Let $\hat{U}(t) = \exp(-i\hat{H}t/\hbar)$ where $t$ is time and $\hat{H}$ is the Hamiltonian. Let $|\psi(0)\rangle$ be the state $t=0$. Show that $|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle$ is the solution to the Time Dependent Schrödinger Equation. That is the state evolves according to a unitary map – explain why this is required by conservation of probability.

(d) Let $\{ |u_n\rangle \}$ be the complete set of energy eigenfunctions, $\hat{H}|u_n\rangle = E_n |u_n\rangle$. Show that $\hat{U}(t) = \sum_n e^{-i\omega_n t} |u_n\rangle \langle u_n|$ where $\hbar \omega_n = E_n$. Using this show that $|\psi(t)\rangle = \sum_n c_n e^{-i\omega_n t} |u_n\rangle$, where $c_n = \langle u_n |\psi(0)\rangle$.

Problem 2: A two-dimensional Hilbert space (20 Points)

Consider a two dimensional Hilbert space spanned by an orthonormal basis $\{|\uparrow\rangle, |\downarrow\rangle\}$. Let us define the operators

$$\hat{S}_x = \frac{\hbar}{2} (|\uparrow\rangle \langle \downarrow| + |\downarrow\rangle \langle \uparrow|), \quad \hat{S}_y = \frac{\hbar}{2i} (|\uparrow\rangle \langle \downarrow| - |\downarrow\rangle \langle \uparrow|), \quad \hat{S}_z = \frac{\hbar}{2} (|\uparrow\rangle \langle \uparrow| - |\downarrow\rangle \langle \downarrow|).$$

(a) Show that each of these operators are Hermitian.

(b) Find the matrix representations of these operators in the basis $\{|\uparrow\rangle, |\downarrow\rangle\}$.
(c) Show that, $[\hat{S}_x, \hat{S}_y] = i\hbar \hat{S}_z$ and cyclic permutations. Do this two way: Using the Dirac notation definition above and the matrix representations you found in (b). Given these commutators, how do you interpret these operators.

Let $|\pm\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \pm |\downarrow\rangle)$

(d) Show that these vectors form a new orthonormal basis.

(e) Find the matrix representations of these operators in the basis $\{|+\rangle, |-\rangle\}$; comment.

(f) The matrices found in (b) and (e) are related through a similarity transformation by a unitary matrix, $U$.

$$
S_x^{(\uparrow\downarrow)} = U^\dagger S_x^{(\pm)} U, \quad S_y^{(\uparrow\downarrow)} = U^\dagger S_y^{(\pm)} U, \quad S_z^{(\uparrow\downarrow)} = U^\dagger S_z^{(\pm)} U,
$$

where the subscript donates the basis in which the operator is represented.

Find $U$ and show that it is unitary.

Now let $\hat{S}_z = (\hat{S}_z \pm i\hat{S}_y)/\hbar$.

(g) Express $\hat{S}_z$ as outer products in the basis $\{|\uparrow\rangle, |\downarrow\rangle\}$ and show that $\hat{S}_z^\dagger = \hat{S}_z$.

(h) Show that $\hat{S}_z |\uparrow\rangle = |\uparrow\rangle$, $\hat{S}_z |\downarrow\rangle = 0$, $\hat{S}_z |\uparrow\rangle = 0$, $\hat{S}_z |\downarrow\rangle = |\downarrow\rangle$ and find $\langle \uparrow | \hat{S}_z^\dagger | \uparrow \rangle$, $\langle \downarrow | \hat{S}_z | \downarrow \rangle$, $\langle \uparrow | \hat{S}_z | \downarrow \rangle$, $\langle \downarrow | \hat{S}_z | \uparrow \rangle$. 