Physics 492: Quantum Mechanics II

Problem Set #1
Due: Wed, Feb. 4, 2004

Problem 1: Unitary operators (10 Points)

An important class of operators are *unitary*, defined as those that *preserve inner product*, i.e. if $|\tilde{\psi}\rangle = \hat{U}|\psi\rangle$ and $|\tilde{\phi}\rangle = \hat{U}|\phi\rangle$, then $\langle\tilde{\phi}|\tilde{\psi}\rangle = \langle\phi|\psi\rangle$ and $\langle\tilde{\psi}|\tilde{\phi}\rangle = \langle\psi|\phi\rangle$.

(a) Show that unitary operators $\hat{U}^\dagger\hat{U} = \hat{U}\hat{U}^\dagger = \hat{1}$ (i.e. the adjoint is the inverse).
(b) Consider $\hat{U} = \exp\left(i\hat{A}\right)$, where $\hat{A}$ is a Hermitian operator. Show that $\hat{U}^\dagger = \exp\left(-i\hat{A}\right)$ and thus show $\hat{U}$ is unitary.
(c) Let $\hat{U}(t) = \exp\left(-i\hat{H}t/\hbar\right)$ where *t* is time and $\hat{H}$ is the Hamiltonian. Let $|\psi(0)\rangle$ be the state *t*=0. Show that $|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle$ is the solution to the Time Dependent Schrödinger Equation. That is the state evolves according to a unitary map – explain why this is *required* by conservation of probability.
(d) Let $\{|u_n\rangle\}$ be the complete set of energy eigenfunctions, $\hat{H}|u_n\rangle = E_n|u_n\rangle$. Show that $\hat{U}(t) = \sum_n e^{-i\omega_n t}|u_n\rangle\langle u_n|$, where $\hbar\omega_n = E_n$. Using this show that $|\psi(t)\rangle = \sum_n c_n e^{-i\omega_n t}|u_n\rangle$, where $c_n = \langle u_n|\psi(0)\rangle$.

Problem 2: A two-dimensional Hilbert space (20 Points)

Consider a two dimensional Hilbert space spanned by an orthonormal basis $\{|\uparrow\rangle, |\downarrow\rangle\}$. Let us define the operators

$$\hat{S}_x = \frac{\hbar}{2}\left(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|\right), \quad \hat{S}_y = \frac{\hbar}{2i}\left(|\uparrow\rangle\langle\downarrow| - |\downarrow\rangle\langle\uparrow|\right), \quad \hat{S}_z = \frac{\hbar}{2}\left(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|\right).$$

(a) Show that each of these operators are Hermitian.
(b) Find the matrix representations of these operators in the basis $\{|\uparrow\rangle, |\downarrow\rangle\}$.

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(c) Show that, $\left[\hat{S}_x, \hat{S}_y\right] = i\hbar\hat{S}_z$ and cyclic permutations. Do this two way: Using the Dirac notation definition above and the matrix representations you found in (b). Given these commutators, how do you interpret these operators.

$$\text{Let } |\pm\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle \pm |\downarrow\rangle\right)$$

(d) Show that these vectors form a new orthonormal basis.

(e) Find the matrix representations of these operators in the basis $\{|+\rangle, |-\rangle\}$; comment.

(f) The matrices found in (b) and (e) are related through a *similarity transformation* by a unitary matrix, U,

$$S_x^{(\uparrow,\downarrow)} = U^\dagger S_x^{(\pm)} U, \quad S_y^{(\uparrow,\downarrow)} = U^\dagger S_y^{(\pm)} U, \quad S_y^{(\uparrow,\downarrow)} = U^\dagger S_y^{(\pm)} U,$$

where the subscript donates the basis in which the operator is represented.

Find U and show that it is unitary.

Now let $\hat{S}_\pm = \left(\hat{S}_x \pm i\hat{S}_y\right)/\hbar$,

(g) Express $\hat{S}_\pm$ as outer products in the basis $\{|\uparrow\rangle, |\downarrow\rangle\}$ and show that $\hat{S}_+^\dagger = \hat{S}_-$.

(h) Show that $\hat{S}_+|\downarrow\rangle = |\uparrow\rangle$, $\hat{S}_+|\uparrow\rangle = 0$, $\hat{S}_-|\downarrow\rangle = 0$, $\hat{S}_-|\uparrow\rangle = |\downarrow\rangle$ and find $\langle\uparrow|\hat{S}_+$, $\langle\downarrow|\hat{S}_+$, $\langle\uparrow|\hat{S}_-$, $\langle\downarrow|\hat{S}_-$.