

Physics 492 - Quantum II

Problem Set #1 - Solutions

Problem 1: Unitary Operators

\hat{U} preserves the inner product, i.e. $\forall |\psi\rangle, |\tilde{\psi}\rangle = \hat{U}|\psi\rangle$
 $\Rightarrow \langle \tilde{\phi} | \tilde{\psi} \rangle = \langle \phi | \psi \rangle$

(a) But $\langle \tilde{\phi} | \tilde{\psi} \rangle = \langle \phi | \hat{U}^\dagger \hat{U} | \psi \rangle = \langle \phi | \psi \rangle$
 $\Rightarrow \boxed{\hat{U}^\dagger \hat{U} = \mathbb{1}}$

$\Rightarrow \hat{U}^\dagger |\tilde{\psi}\rangle = \hat{U}^\dagger \hat{U} |\psi\rangle = \mathbb{1} |\psi\rangle = |\psi\rangle$

$\therefore \langle \phi | \psi \rangle = \langle \tilde{\phi} | \hat{U} \hat{U}^\dagger | \tilde{\psi} \rangle = \langle \tilde{\phi} | \tilde{\psi} \rangle$
 $\Rightarrow \boxed{\hat{U} \hat{U}^\dagger = \mathbb{1}}$

Thus for unitary operators, the inverse is the identity

(b) Let $\hat{U} = \exp(i\hat{A})$ where \hat{A} is Hermitian
(i.e. $\hat{A}^\dagger = \hat{A}$)

$\hat{U} = \sum_{n=0}^{\infty} \frac{1}{n!} (i\hat{A})^n$

$\hat{U}^\dagger = \sum_{n=0}^{\infty} \frac{1}{n!} [(i\hat{A})^n]^\dagger = \sum_{n=0}^{\infty} \frac{1}{n!} [(-i\hat{A}^\dagger)]^n$

$\Rightarrow \hat{U}^\dagger = \sum_{n=0}^{\infty} \frac{1}{n!} (-i\hat{A})^n = \boxed{\exp(-i\hat{A})}$

zero operator

$$\text{thus } \hat{U}^\dagger \hat{U} = e^{-i\hat{A}} e^{i\hat{A}} = e^{-i\hat{A} + i\hat{A}} = e^{\hat{0}}$$

$$= \hat{1} = \hat{U} \hat{U}^\dagger \Rightarrow \hat{U} \text{ is unitary}$$

(Aside: Note in general $e^{\hat{A}} e^{\hat{B}} \neq e^{\hat{A} + \hat{B}}$
 unless \hat{A} and \hat{B} commute)

(c) Let $\hat{U}(t) = e^{-i\hat{H}t/\hbar}$ where \hat{H} = Hamiltonian

Given a state at $t=0$ $|\psi(0)\rangle$

Let $|\psi(t)\rangle = \hat{U}(t) |\psi(0)\rangle$

$$\Rightarrow \frac{\partial}{\partial t} |\psi(t)\rangle = \frac{\partial \hat{U}(t)}{\partial t} |\psi(0)\rangle$$

(Aside $\frac{\partial \hat{U}(t)}{\partial t} = \frac{\partial}{\partial t} e^{-i\hat{H}t/\hbar} = -\frac{i\hat{H}}{\hbar} e^{-i\hat{H}t/\hbar}$
 $= -\frac{i\hat{H}}{\hbar} \hat{U}(t)$)

$$\Rightarrow \frac{\partial}{\partial t} |\psi(t)\rangle = -\frac{i\hat{H}}{\hbar} \hat{U}(t) |\psi(0)\rangle = -\frac{i\hat{H}}{\hbar} |\psi(t)\rangle$$

$$\Rightarrow \boxed{\frac{\hbar}{-i} \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle}$$

thus $|\psi(t)\rangle = \hat{U}(t) |\psi(0)\rangle$ is a solution
 to the Time Dependent Schrödinger Eqn

That $|\psi\rangle$ evolve according to unitary map is required by the interpretation of $|\psi\rangle$

At $|\psi(0)\rangle$ we choose $\|\psi(0)\|^2 = \langle\psi(0)|\psi(0)\rangle = 1$, normalized to a total probability of all possible alternatives. At a later time this total probability must be conserved

$$\Rightarrow \|\psi(t)\|^2 = 1 \Rightarrow \langle\psi(t)|\psi(t)\rangle = \langle\psi(0)|\psi(0)\rangle$$

Unitary time evolution

(d) Let $\{|u_n\rangle\}$ satisfy $\hat{A}|u_n\rangle = E_n|u_n\rangle$
 \uparrow complete set of orthonormal functions

$$\Rightarrow \hat{1} = \sum_n |u_n\rangle\langle u_n|$$

$$\hat{U} = \hat{U} \sum_n |u_n\rangle\langle u_n| = \sum_n e^{-i\hat{A}t/\hbar} |u_n\rangle\langle u_n|$$

$$\Rightarrow \hat{U}(t) = \sum_n e^{-iE_n t/\hbar} |u_n\rangle\langle u_n| = \sum_n e^{-i\omega_n t/\hbar} |u_n\rangle\langle u_n|$$

where $\hbar\omega_n = E_n$

Thus, $|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle = \sum_n e^{-i\omega_n t/\hbar} |u_n\rangle \underbrace{\langle u_n|\psi(0)\rangle}_{= C_n}$

$$\Rightarrow |\psi(t)\rangle = \sum_n e^{-i\omega_n t/\hbar} C_n |u_n\rangle$$

where $C_n = \langle u_n|\psi(0)\rangle$

Problem 2: A two-dimensional Hilbert Space

Orthonormal basis: $\{|\uparrow\rangle, |\downarrow\rangle\}$ $\langle\uparrow|\uparrow\rangle = \langle\downarrow|\downarrow\rangle = 1$
 $\langle\uparrow|\downarrow\rangle = \langle\downarrow|\uparrow\rangle = 0$

$$\hat{S}_x = \frac{\hbar}{2} (|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|)$$

$$\hat{S}_y = \frac{\hbar}{2i} (|\uparrow\rangle\langle\downarrow| - |\downarrow\rangle\langle\uparrow|)$$

$$\hat{S}_z = \frac{\hbar}{2} (|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|)$$

(a) Recall $(|\psi\rangle\langle\phi|)^\dagger = |\phi\rangle\langle\psi|$

Thus, by inspection $\hat{S}_x^\dagger = \hat{S}_x$ ✓ and $\hat{S}_z^\dagger = \hat{S}_z$ ✓

$$\hat{S}_y^\dagger = \frac{\hbar}{-2i} (|\downarrow\rangle\langle\uparrow| - |\uparrow\rangle\langle\downarrow|) = \frac{\hbar}{2i} (|\uparrow\rangle\langle\downarrow| - |\downarrow\rangle\langle\uparrow|)$$

$$= \hat{S}_y \quad \checkmark \quad \text{all are Hermitian}$$

(b) Matrix representation in basis $\{|\uparrow\rangle, |\downarrow\rangle\}$

$$\hat{S}_x = \begin{matrix} \left[\begin{array}{cc} \langle\uparrow|\hat{S}_x|\uparrow\rangle & \langle\uparrow|\hat{S}_x|\downarrow\rangle \\ \langle\downarrow|\hat{S}_x|\uparrow\rangle & \langle\downarrow|\hat{S}_x|\downarrow\rangle \end{array} \right] \cdot \left. \begin{array}{l} \langle\uparrow| \\ \langle\downarrow| \end{array} \right\} \text{rows} \\ \begin{matrix} \text{column: } |\uparrow\rangle & |\downarrow\rangle \end{matrix} \end{matrix}$$

$$\langle\uparrow|\hat{S}_x|\uparrow\rangle = \frac{\hbar}{2} (\langle\uparrow|\uparrow\rangle\langle\downarrow|\uparrow\rangle + \langle\uparrow|\downarrow\rangle\langle\uparrow|\uparrow\rangle) = 0$$

$$\langle\downarrow|\hat{S}_x|\downarrow\rangle = \frac{\hbar}{2} (\langle\downarrow|\uparrow\rangle\langle\downarrow|\downarrow\rangle + \langle\downarrow|\downarrow\rangle\langle\uparrow|\downarrow\rangle) = 0$$

$$\langle\uparrow|\hat{S}_x|\downarrow\rangle = \frac{\hbar}{2} (\underbrace{\langle\uparrow|\uparrow\rangle}_{=1}\langle\downarrow|\downarrow\rangle + \underbrace{\langle\uparrow|\downarrow\rangle}_0 \underbrace{\langle\uparrow|\uparrow\rangle}_0) = 1$$

$$\langle \downarrow | \hat{S}_x | \uparrow \rangle = \langle \uparrow | \hat{S}_x | \downarrow \rangle^* \quad (\text{since } \hat{S}_x \text{ Hermitian})$$

$$= \frac{\hbar}{2}$$

$$\Rightarrow \boxed{\hat{S}_x = \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$$

Similarly $\hat{S}_y = \begin{bmatrix} \langle \uparrow | \hat{S}_y | \uparrow \rangle & \langle \uparrow | \hat{S}_y | \downarrow \rangle \\ \langle \downarrow | \hat{S}_y | \uparrow \rangle & \langle \downarrow | \hat{S}_y | \downarrow \rangle \end{bmatrix}$

$$\Rightarrow \boxed{\hat{S}_y = \frac{\hbar}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}}$$

$$\hat{S}_z = \begin{bmatrix} \langle \uparrow | \hat{S}_z | \uparrow \rangle & \langle \uparrow | \hat{S}_z | \downarrow \rangle \\ \langle \downarrow | \hat{S}_z | \uparrow \rangle & \langle \downarrow | \hat{S}_z | \downarrow \rangle \end{bmatrix}$$

$$\Rightarrow \boxed{\hat{S}_z = \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}$$

$$(c) \quad [\hat{S}_x, \hat{S}_y] = \hat{S}_x \hat{S}_y - \hat{S}_y \hat{S}_x$$

$$= \frac{\hbar^2}{4i} \left[(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|) (|\uparrow\rangle\langle\downarrow| - |\downarrow\rangle\langle\uparrow|) \right.$$

$$\left. - (|\uparrow\rangle\langle\downarrow| - |\downarrow\rangle\langle\uparrow|) (|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|) \right]$$

$$= \frac{\hbar^2}{4i} (|\downarrow\rangle\langle\downarrow| - |\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow| - |\uparrow\rangle\langle\uparrow|)$$

Thus,
$$[\hat{S}_x, \hat{S}_y] = \frac{\hbar^2}{4i} (2|\downarrow\rangle\langle\downarrow| - 2|\uparrow\rangle\langle\uparrow|)$$

$$= i\hbar \frac{\hbar}{2} (|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|)$$

$$\Rightarrow \boxed{[\hat{S}_x, \hat{S}_y] = i\hbar \hat{S}_z}$$

Cyclic permutation: $[\hat{S}_z, \hat{S}_x] = i\hbar \hat{S}_y$?

$$\hat{S}_z \hat{S}_x = \frac{\hbar^2}{4} (|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|) (|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|)$$

$$= \frac{\hbar^2}{4} (|\uparrow\rangle\langle\downarrow| - |\downarrow\rangle\langle\uparrow|)$$

$$\hat{S}_x \hat{S}_z = \frac{\hbar^2}{4} (|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|) (|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|)$$

$$= \frac{\hbar^2}{4} (|\downarrow\rangle\langle\uparrow| - |\uparrow\rangle\langle\downarrow|)$$

$$\Rightarrow \hat{S}_z \hat{S}_x - \hat{S}_x \hat{S}_z = \frac{\hbar^2}{2} (|\uparrow\rangle\langle\downarrow| - |\downarrow\rangle\langle\uparrow|)$$

$$= i\hbar \frac{\hbar}{2i} (|\uparrow\rangle\langle\downarrow| - |\downarrow\rangle\langle\uparrow|)$$

$$\Rightarrow \boxed{[\hat{S}_z, \hat{S}_x] = i\hbar \hat{S}_y}$$

Similarly, the same procedure yields

$$\boxed{[\hat{S}_y, \hat{S}_z] = i\hbar \hat{S}_x}$$

For comparison, let's carry out the same calculation using the matrix representations

$$\hat{S}_x \hat{S}_y = \frac{\hbar^2}{4} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \frac{\hbar^2}{4} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$$\hat{S}_y \hat{S}_x = \frac{\hbar^2}{4} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \frac{\hbar^2}{4} \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$$

$$\Rightarrow \hat{S}_x \hat{S}_y - \hat{S}_y \hat{S}_x = \frac{\hbar^2}{2} \begin{bmatrix} +i & 0 \\ 0 & -i \end{bmatrix} = i\hbar \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow [\hat{S}_x, \hat{S}_y] = i\hbar \hat{S}_z \quad \checkmark$$

We recognize these commutation relations as the same as those for the components of the angular momentum vector $\hat{L} = \hat{r} \times \hat{p}$

$$[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z, \text{ and cycle perms.}$$

We will see that these operators represent the components of spin angular momentum

for a spin- $\frac{1}{2}$ particle.

Now let $|\pm\rangle \equiv \frac{1}{\sqrt{2}}(|\uparrow\rangle \pm |\downarrow\rangle)$

$$\begin{aligned} \langle +|+ \rangle &= \frac{1}{2} (\langle \uparrow| + \langle \downarrow|) (|\uparrow\rangle + |\downarrow\rangle) \\ &= \frac{1}{2} (\underbrace{\langle \uparrow|\uparrow \rangle}_1 + \underbrace{\langle \downarrow|\uparrow \rangle + \langle \uparrow|\downarrow \rangle}_{=0} + \underbrace{\langle \downarrow|\downarrow \rangle}_1) \end{aligned}$$

$$\Rightarrow \boxed{\langle +|+ \rangle = 1} \quad \checkmark$$

$$\begin{aligned} \langle -|- \rangle &= \frac{1}{2} (\langle \uparrow| - \langle \downarrow|) (|\uparrow\rangle - |\downarrow\rangle) \\ &= \frac{1}{2} (\underbrace{\langle \uparrow|\uparrow \rangle}_1 - \underbrace{\langle \downarrow|\uparrow \rangle + \langle \uparrow|\downarrow \rangle}_0 + \underbrace{\langle \downarrow|\downarrow \rangle}_1) \end{aligned}$$

$$\Rightarrow \boxed{\langle -|- \rangle = 1} \quad \checkmark$$

$$\begin{aligned} \langle +|- \rangle &= \frac{1}{2} (\langle \uparrow| + \langle \downarrow|) (|\uparrow\rangle - |\downarrow\rangle) \\ &= \frac{1}{2} (\underbrace{\langle \uparrow|\uparrow \rangle}_1 + \underbrace{\langle \downarrow|\uparrow \rangle}_{\downarrow} - \underbrace{\langle \uparrow|\downarrow \rangle}_{\downarrow} - \underbrace{\langle \downarrow|\downarrow \rangle}_1) \end{aligned}$$

$$\Rightarrow \boxed{\langle +|- \rangle = \langle -|+ \rangle^* = 0}$$

$\Rightarrow \{ |+\rangle, |-\rangle \}$ is another orthonormal basis

(e) In this basis,

$$\hat{S}_x = \begin{bmatrix} \langle +|\hat{S}_x|+ \rangle & \langle +|\hat{S}_x|- \rangle \\ \langle -|\hat{S}_x|+ \rangle & \langle -|\hat{S}_x|- \rangle \end{bmatrix}$$

$$\langle +|\hat{S}_x|+ \rangle = \frac{\hbar}{2} \left(\underbrace{\langle +|\uparrow \rangle}_{\frac{1}{\sqrt{2}}} \underbrace{\langle \downarrow|+ \rangle}_{\frac{1}{\sqrt{2}}} + \underbrace{\langle +|\downarrow \rangle}_{\frac{1}{\sqrt{2}}} \underbrace{\langle \uparrow|+ \rangle}_{\frac{1}{\sqrt{2}}} \right) = \frac{\hbar}{2}$$

$$\langle +|\hat{S}_x|- \rangle = \frac{\hbar}{2} \left(\underbrace{\langle +|\uparrow \rangle}_{+\frac{1}{\sqrt{2}}} \underbrace{\langle \downarrow|- \rangle}_{-\frac{1}{\sqrt{2}}} + \underbrace{\langle +|\downarrow \rangle}_{+\frac{1}{\sqrt{2}}} \underbrace{\langle \uparrow|- \rangle}_{+\frac{1}{\sqrt{2}}} \right) = 0$$

$$= \langle -|\hat{S}_x|+ \rangle^* \quad (\text{since } \hat{S}_x \text{ Hermitian})$$

$$\langle -|\hat{S}_x|- \rangle = \frac{\hbar}{2} \left(\underbrace{\langle -|\uparrow \rangle}_{+\frac{1}{\sqrt{2}}} \underbrace{\langle \downarrow|- \rangle}_{-\frac{1}{\sqrt{2}}} + \underbrace{\langle -|\downarrow \rangle}_{-\frac{1}{\sqrt{2}}} \underbrace{\langle \uparrow|- \rangle}_{+\frac{1}{\sqrt{2}}} \right) = -\frac{\hbar}{2}$$

$$\Rightarrow \hat{S}_x^{(+,-)} = \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\hat{S}_y = \begin{bmatrix} \langle +|\hat{S}_y|+ \rangle & \langle +|\hat{S}_y|- \rangle \\ \langle -|\hat{S}_y|+ \rangle & \langle -|\hat{S}_y|- \rangle \end{bmatrix}$$

$$\langle +|\hat{S}_y|+ \rangle = \frac{\hbar}{2i} \left(\langle +|\uparrow \rangle \langle \downarrow|+ \rangle - \langle +|\downarrow \rangle \langle \uparrow|+ \rangle \right) = 0$$

$$\begin{aligned} \langle +|\hat{S}_y|- \rangle &= \frac{\hbar}{2i} \left(\langle +|\uparrow \rangle \langle \downarrow|- \rangle - \langle +|\uparrow \rangle \langle \downarrow|- \rangle \right) = -\frac{\hbar}{2i} \\ &= \langle -|\hat{S}_y|+ \rangle^* \end{aligned}$$

$$\langle -1 | \hat{S}_y | - \rangle = \frac{\hbar}{2i} (\langle -1 | \uparrow \rangle \langle \downarrow | - \rangle - \langle -1 | \downarrow \rangle \langle \uparrow | - \rangle) = 0$$

$$\hat{S}_y^{(+,-)} = \frac{\hbar}{2} \begin{bmatrix} 0 & +i \\ -i & 0 \end{bmatrix}$$

$$\hat{S}_z^{(+,-)} = \begin{bmatrix} \langle +1 | \hat{S}_z | + \rangle & \langle +1 | \hat{S}_z | - \rangle \\ \langle -1 | \hat{S}_z | + \rangle & \langle -1 | \hat{S}_z | - \rangle \end{bmatrix}$$

$$\langle +1 | \hat{S}_z | + \rangle = \frac{\hbar}{2} (|\langle +1 | \uparrow \rangle|^2 - |\langle +1 | \downarrow \rangle|^2) = 0$$

$$\begin{aligned} \langle +1 | \hat{S}_z | - \rangle &= \frac{\hbar}{2} (\underbrace{\langle +1 | \uparrow \rangle \langle \uparrow | - \rangle}_{\frac{1}{2}} - \underbrace{\langle +1 | \downarrow \rangle \langle \downarrow | - \rangle}_{-\frac{1}{2}}) = \frac{\hbar}{2} \\ &= \langle -1 | \hat{S}_z | + \rangle^* \end{aligned}$$

$$\langle -1 | \hat{S}_z | - \rangle = \frac{\hbar}{2} (|\langle -1 | \uparrow \rangle|^2 - |\langle -1 | \downarrow \rangle|^2) = 0$$

$$\Rightarrow \hat{S}_z^{(+,-)} = \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Note in this basis \hat{S}_x is diagonal, whereas in the $\{|\uparrow\rangle, |\downarrow\rangle\}$ basis \hat{S}_z is diagonal.

More on this later...

$$(f) \quad \text{Let } |e_1\rangle = |\uparrow\rangle \quad |e_2\rangle = |\downarrow\rangle \\ |e'_1\rangle = |+\rangle \quad |e'_2\rangle = |-\rangle$$

Let $S_{ij} = \langle e_i | \hat{S} | e_j \rangle$ For any component $\hat{S}_x, \hat{S}_y, \hat{S}_z$
 \uparrow Matrix elements in $\{|\uparrow\rangle, |\downarrow\rangle\}$ basis

We want to relate these matrix elements to those in the $\{|+\rangle, |-\rangle\}$. To do this simply just insert a complete set

$$\sum_k |e'_k\rangle \langle e'_k| = \hat{1}$$

$$S_{ij} = \langle e_i | \hat{1} \hat{S} \hat{1} | e_j \rangle = \sum_{k,l} \langle e_i | e'_k \rangle \langle e'_k | \hat{S} | e'_l \rangle \langle e'_l | e_j \rangle$$

Now $\tilde{S}_{kl} \equiv \langle e'_k | \hat{S} | e'_l \rangle =$ matrix elements in $\{|+\rangle, |-\rangle\}$

Let $U_{lj} = \langle e'_l | e_j \rangle$ these form the elements of a matrix

$$\hat{U} \equiv \begin{bmatrix} \langle e'_1 | e_1 \rangle & \langle e'_1 | e_2 \rangle \\ \langle e'_2 | e_1 \rangle & \langle e'_2 | e_2 \rangle \end{bmatrix} = \begin{bmatrix} \langle + | \uparrow \rangle & \langle + | \downarrow \rangle \\ \langle - | \uparrow \rangle & \langle - | \downarrow \rangle \end{bmatrix}$$

Note $\langle e_i | e'_k \rangle = \langle e'_k | e_i \rangle^* = (U_{ki})^* = U_{ik}^\dagger$

Thus $S_{ij} = \sum_{k,l} (U^\dagger)_{ik} \zeta_{kl} U_{lj}$

$$S = U^\dagger \zeta U \quad (\text{matrix multiplication})$$

Check: Is U unitary?

Using Dirac notation: $(U^\dagger U)_{ij} = \sum_k (U^\dagger)_{ik} U_{kj}$

$$(U^\dagger U)_{ij} = \sum_k \langle e_i | e_k \rangle \langle e_k | e_j \rangle = \langle e_i | \underbrace{\sum_k |e_k\rangle \langle e_k|}_{= \mathbb{1}} | e_j \rangle$$
$$= \delta_{ij}$$

$$\Rightarrow U^\dagger U = \mathbb{1} \quad \checkmark$$

Using matrices $\hat{U} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$$\hat{U}^\dagger = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad (\hat{U}^\dagger = \hat{U} \text{ and } \hat{U} \text{ real})$$

$$\hat{U}^\dagger \hat{U} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

These U is unitary

$$\tilde{S}_x = \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{aligned} U^\dagger \tilde{S}_x U &= \frac{\hbar}{4} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \\ &= \frac{\hbar}{4} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{\hbar}{4} \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \end{aligned}$$

$$\Rightarrow U^\dagger \tilde{S}_x U = S_x \quad \checkmark$$

$$\tilde{S}_y = \frac{\hbar}{2i} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\begin{aligned} U^\dagger \tilde{S}_y U &= \frac{\hbar}{4i} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \\ &= \frac{\hbar}{4i} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \frac{\hbar}{4i} \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \end{aligned}$$

$$\Rightarrow U^\dagger \tilde{S}_y U = \frac{\hbar}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \checkmark$$

$$\tilde{S}_z = \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{aligned} U^\dagger \tilde{S}_z U &= \frac{\hbar}{4} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \\ &= \frac{\hbar}{4} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{\hbar}{4} \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \end{aligned}$$

$$\Rightarrow U^\dagger \tilde{S}_z U = \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \checkmark$$

Now let $\hat{S}_{\pm} = \frac{\hat{S}_x \pm i \hat{S}_y}{\hbar}$

$$(g) \Rightarrow \hat{S}_{\pm} = \frac{1}{2}(\uparrow\downarrow\langle\downarrow + \downarrow\downarrow\langle\uparrow) \pm \frac{1}{2}(\uparrow\uparrow\langle\downarrow - \downarrow\downarrow\langle\uparrow)$$

$$\Rightarrow \boxed{\hat{S}_+ = \uparrow\downarrow\langle\downarrow} \quad \boxed{\hat{S}_- = \downarrow\downarrow\langle\uparrow}$$

By inspection $\hat{S}_+^\dagger = \hat{S}_-$

$$(h) \quad \hat{S}_+ \downarrow\downarrow = \uparrow\uparrow \quad \hat{S}_+ \uparrow\uparrow = 0 \quad (\text{null vector})$$

$$\hat{S}_- \uparrow\uparrow = \downarrow\downarrow \quad \hat{S}_- \downarrow\downarrow = 0$$

Thus \hat{S}_{\pm} are "raising and lowering operators"

$$\text{Finally, } \langle\uparrow|\hat{S}_+ = (\hat{S}_- \uparrow\uparrow)^\dagger = \downarrow\downarrow$$

$$\langle\downarrow|\hat{S}_+ = (\hat{S}_- \downarrow\downarrow)^\dagger = 0$$

$$\langle\uparrow|\hat{S}_- = (\hat{S}_+ \uparrow\uparrow)^\dagger = 0$$

$$\langle\downarrow|\hat{S}_- = (\hat{S}_+ \downarrow\downarrow)^\dagger = \langle\uparrow$$

$$\text{Note } \hat{S}_+ \doteq \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \hat{S}_- \doteq \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

in this basis