Physics 492: Quantum Mechanics II

Problem Set #2
Due: Wed, Feb. 11, 2004

Problem 1: More on operator algebra and Dirac notation. (10 Points)

(a) Show that $\left(\hat{A}\hat{B}\right)^{\dagger} = \hat{B}^{\dagger}\hat{A}^{\dagger}$.

(b) Suppose we have a Hermitian operator $\hat{A}$ with eigenvalues/vectors $\hat{A}|a\rangle = a|a\rangle$. Given a arbitrary normalized state $|\psi\rangle = \sum_a c_a |a\rangle$; the probability of measuring a is $P_a = |c_a|^2$. The average value, or expectation value, is then defined, $\langle \hat{A} \rangle = \sum_a a\, P_a$.

Show that $\langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle$.

(c) Suppose $\hat{A}$ is a *positive operator*. Show that all of its eigenvalues are nonnegative.

(d) The n^{th} excited state the harmonic oscillator is defined in terms of the ground state by $|n\rangle = \frac{1}{\sqrt{n!}}\left(\hat{a}^{\dagger}\right)^n |0\rangle$, If $|0\rangle$ is normalized, show that $|n\rangle$ is normalized.

Problem 2: Physical examples of harmonic oscillators. (10 Points)
No potential is harmonic for arbitrarily large displacements from the origin. Eventually nonlinearities set in. But, as long as the displacement is small, so that the lowest order quadratic term dominates, we can treat the potential as harmonic. However, there is zero point motion of the quantum state. If the extent of the ground state is large compared to the range over which the potential is quadratic, the spectrum in no way look like that of an SHO.

Consider the following potentials. Find the characteristic oscillation frequency near an equilibrium. Determine the extent of the ground state for the corresponding SHO. By comparing this to the characteristic scale over which the potential is harmonic, estimate the number of levels for which the energy spectrum looks harmonic (i.e. equally space). Sketch the energy level diagram on top of the potential for some choice of the parameters.

(a) A periodic potential: $V(x) = V_0 \sin^2(kx)$.

(b) A Leonard-Jones potential binding a diatomic molecule: $V(r) = \frac{C_{12}}{r^{12}} - \frac{C_6}{r^6}$.

(c) The effective potential for radial motion of the electron in hydrogen in a p-state:

$$V(r) = \frac{\hbar^2}{mr^2} - \frac{e^2}{r}.$$

Problem 3: Liboff 7.16. (10 Points)