Physics 492: Quantum Mechanics II
Problem Set #3
Due: Wed, Feb. 18, 2004

Problem 3: The Coherent State (30 Points)

Consider a wavepacket describing a mass *m* in a harmonic potential of frequency ω consisting of a superposition of energy eigenstates defined as

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle, \quad \text{where } c_n = e^{-|\alpha|^2/2} \frac{(\alpha)^n}{\sqrt{n!}}, \text{ with } \alpha \text{ a complex number.}$$

Such a state is know as a "coherent state" or sometime a "quasi-classical state.

(a) Show that $|\alpha\rangle$ is an eigenstate of the annihilation operator, $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$.
(b) Show that in this state $\langle \hat{x} \rangle = x_c \operatorname{Re}(\alpha)$, $\langle \hat{p} \rangle = p_c \operatorname{Im}(\alpha)$, where $x_c = \sqrt{2\hbar / m\omega}$.

One can show that, in position space, the wave function for this state is

$$\psi_\alpha(x) = e^{ip_0 x/\hbar} u_0(x - x_0)$$

where $u_0(x)$ is the ground state Gaussian wavepacket and $x_0 = x_c \operatorname{Re}(\alpha)$, $p_0 = p_c \operatorname{Im}(\alpha)$.

(c) What is the wave function in momentum space? Interpret x_0 and p_0.
(d) Explicitly show that $\psi_a(x)$ is an eigenstate of the annihilation operator using its position-space representation of the annihilation operator.
(e) Show that the coherent state is a minimum uncertainty wavepacket (with equal uncertainties in *x* and *p*, in characteristic dimensionless units).
(f) If at time *t*=0 the state is $|\psi(0)\rangle = |\alpha\rangle$, show that at a later time,

$$|\psi(t)\rangle = e^{-i\omega t/2} \left| \alpha e^{-i\omega t} \right\rangle \text{ --interpret.}$$

(g) Show that, as a function of time, $\langle \hat{x} \rangle$ and $\langle \hat{p} \rangle$ follow the *classical trajectory* of the harmonic oscillator, hence the name "quasiclassical state".
(h) Write the wave function as a function of time, $\psi_a(x,t)$. Sketch a cartoon of the time evolving probability density.

(i) Show that $\langle \hat{N} \rangle = |\alpha|^2$ and $\Delta N = |\alpha| = \sqrt{\langle N \rangle}$, so that in the classical limit $\lim_{|\alpha| \to \infty} \Delta N / \langle N \rangle \to 0$.
(j) The statistics associated with the number of excitations found in (i) are those of "counting statistics", i.e. a Poisson distribution, discussed last semester. Show that the probability distribution in *n* is indeed Poissonian, with the appropriate parameters.
(k) The phase of a classical oscillator is $\phi(t) = \omega t - \phi_0$. Use the "rough" time-energy uncertainty principle $\Delta E \Delta t > \hbar$, to find an uncertainty principle between the number and phase of a quantum oscillator.