Physic 492: Quantum II

Problem Set #3 Solutions

Problem - Coherent State of the SHO

Definition $|\alpha\rangle \equiv \sum_{n=0}^{\infty} c_n |n\rangle$ (not stationary state)

where $c_n = e^{-\frac{|\alpha|^2}{2}} \frac{\alpha^n}{\sqrt{n!}}$

(a) $\hat{a}|\alpha\rangle = \sum_{n=0}^{\infty} c_n \hat{a}|n\rangle = \sum_{n=0}^{\infty} c_n \sqrt{n}\, |n-1\rangle$

$$= \sum_{n=0}^{\infty} c_{n+1} \sqrt{n+1}\, |n\rangle$$

(Aside: $c_{n+1}\sqrt{n+1} = e^{-\frac{|\alpha|^2}{2}} \frac{\alpha^{n+1}\sqrt{n+1}}{\sqrt{(n+1)!}} = e^{-|\alpha|^2/2} \frac{\alpha^{n+1}}{\sqrt{n!}}$

$= \alpha c_n$)

$$\Rightarrow \hat{a}|\alpha\rangle = \alpha \sum_{n=0}^{\infty} c_n |n\rangle = \alpha |\alpha\rangle$$

So the coherent state is an eigenstate of the annihilation operator.

(α complex #)

(b) Recall $\hat{x} = x_c \hat{X} = x_c \left(\frac{\hat{a} + \hat{a}^\dagger}{2} \right)$ $x_c = \sqrt{2\hbar/m\omega}$

$\hat{p} = p_c \hat{P} = p_c \left(\frac{\hat{a} - \hat{a}^\dagger}{2i} \right)$ $p_c = \sqrt{2m\hbar\omega}$

Also $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ and $\langle\alpha|\hat{a}^\dagger = \alpha^* \langle\alpha|$

$\Rightarrow \langle\alpha|\hat{a}|\alpha\rangle = \alpha \langle\alpha|\alpha\rangle = \alpha$ ($|\alpha\rangle$ is normalized)

$\langle\alpha|\hat{a}^\dagger|\alpha\rangle = \alpha^* \langle\alpha|\alpha\rangle = \alpha^*$

$\Rightarrow \langle\alpha|\hat{x}|\alpha\rangle = x_c \left(\frac{\alpha + \alpha^*}{2} \right) = x_c \, \mathrm{Re}(\alpha)$

$\langle\alpha|\hat{p}|\alpha\rangle = p_c \left(\frac{\alpha - \alpha^*}{2i} \right) = p_c \, \mathrm{Im}(\alpha)$

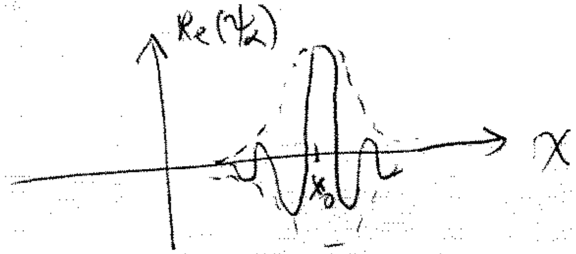
(c)

Given $\langle x|\alpha\rangle = e^{i p_0 x/\hbar} u_0(x - x_0) = \psi_\alpha(x)$

and $\alpha = \frac{x_0}{x_c} + i \frac{p_0}{x_c}$

We see that x_0 + p_0 are the mean position and momentum found in part (b).

$\psi_\alpha(x)$ is a wave packet, Gaussian, centered at $x = x_0$ with "carrier wave" momentum p_0



The momentum space wave function is the Fourier transform of $\psi_\alpha(x)$

$$\tilde{\psi}_\alpha(p) = \int \frac{dx}{\sqrt{2\pi\hbar}} \, \psi_\alpha(x) \, e^{-ipx/\hbar}$$

We can use the convolution theorm

$$\tilde{\psi}_\alpha(p) = \underbrace{\tilde{\mathcal{F}}\left[e^{ip_0x/\hbar}\right]}_{\overset{\parallel}{\delta(p-p_0)}} \circledast \tilde{\mathcal{F}}\left[u_0(x-x_0)\right]$$

Delta function $\rightarrow \delta(p-p_0) \underset{\substack{\uparrow \\ \text{convolution}}}{\circledast} \underbrace{e^{-ix_0p/\hbar}\,\tilde{u}_0(p)}_{\text{Shift theore}}$

$$\Rightarrow \boxed{\tilde{\psi}_\alpha(p) = \underset{\substack{\uparrow \\ \text{Overall} \\ \text{constant phase}}}{e^{-\frac{ix_0p_0}{\hbar}}} \left(e^{-ix_0p/\hbar} \, \tilde{u}_0(p-p_0)\right)}$$

In momentum space, $\tilde{u}_0$ centred at p_0.

the mean position appears as a phase in momentum space.

(d) We seek to show $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ in position space:

$$\langle x|\hat{a}|\alpha\rangle = \langle x|\hat{X}+i\hat{P}|\alpha\rangle = \left(\frac{x}{x_c} + \frac{i}{p_c}\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\psi_\alpha(x)$$

$$= \left(\frac{x}{x_c} + \frac{\hbar}{p_c}\frac{\partial}{\partial x}\right) e^{ip_0x/\hbar} u_0(x-x_0)$$

$$= \frac{x}{x_c} e^{ip_0x/\hbar} u_0(x-x_0) + i\frac{p_0}{p_c} e^{ip_0x/\hbar} u_0(x-x_0)$$

$$+ \frac{\hbar}{p_c} e^{ip_0x/\hbar} \frac{\partial}{\partial x} u_0(x-x_0)$$

$$= e^{ip_0x/\hbar} \underbrace{\left(\frac{x}{x_c} + \frac{\hbar}{p_c}\frac{\partial}{\partial x}\right) u_0(x-x_0)}_{\text{let } y = x - x_0} + i\frac{p_0}{p_c}\psi_\alpha(x)$$

$$\Rightarrow \left(\frac{x_0}{x_c} + \underbrace{\frac{y}{x_c} + \frac{\hbar}{p_c}\frac{\partial}{\partial y}\right) u_0(y)}_{=0}$$

$\hat{a}\, u_0(y) = 0$

$$\therefore \langle x|\hat{a}|\alpha\rangle = \left(\frac{x_0}{x_c} + i\frac{p_0}{p_c}\right)\psi_\alpha(x)$$

$$= \alpha\,\psi_\alpha(x) \checkmark$$

(e) Uncertainties: $\Delta x = \sqrt{\Delta x^2}$, $\Delta p = \sqrt{\Delta p^2}$

$$\Delta x^2 = \langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2, \quad \Delta p^2 = \langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2$$

We have already found $\begin{cases} \langle \hat{x} \rangle = x_c \, \mathrm{Re}(\alpha) \\ \langle \hat{p} \rangle = p_c \, \mathrm{Im}(\alpha) \end{cases}$

$$\langle \hat{x}^2 \rangle = \frac{x_c^2}{4} \langle \alpha | \left(\hat{a}^2 + \hat{a}^{+2} + \hat{a}^{+}\hat{a} + \hat{a}\hat{a}^{+} \right) | \alpha \rangle$$

Use $\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$, $\langle \alpha | \hat{a}^{+} = \alpha^* \langle \alpha |$

$$\Rightarrow \langle \hat{x}^2 \rangle = \frac{x_c^2}{4} \Big(\alpha^2 + \alpha^{*2} + \alpha^* \alpha + \underbrace{\langle \alpha | \hat{a}\hat{a}^{+} | \alpha \rangle}_{= \, \langle \alpha | \hat{a}^{+}\hat{a} + 1 | \alpha \rangle \, = \, \alpha^*\alpha + 1} \Big)$$

$$\Rightarrow \langle \hat{x}^2 \rangle = \frac{x_c^2}{4} (\alpha + \alpha^*)^2 + \frac{x_c^2}{4}$$

$$= (x_c \, \mathrm{Re}(\alpha))^2 + \frac{x_c^2}{4}$$

$$\Rightarrow \Delta x^2 = \frac{x_c^2}{4} \Rightarrow \boxed{\Delta x = \frac{x_c}{2} = \sqrt{\frac{\hbar}{2m\omega}}}$$

Similarly $\Delta p^2 = \frac{p_c}{4} \Rightarrow \boxed{\Delta p = \frac{p_c}{2} = \sqrt{\frac{m\hbar\omega}{2}}}$

$$\boxed{\Delta x \, \Delta p = \frac{\hbar}{2}}$$ minimum uncertain wave packet

(f) at $t=0$ $\quad |\psi(0)\rangle = |\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$

$$|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle = \sum_{n=0}^{\infty} c_n \, \hat{U}(t)|n\rangle$$

$$= \sum_{n=0}^{\infty} c_n \, e^{-iE_n t/\hbar} |n\rangle \qquad \text{where } E_n = \hbar\omega(n+\tfrac{1}{2})$$

$$= e^{-i\omega t/2} \sum_{n=0}^{\infty} c_n e^{-in\omega t} |n\rangle$$

where $c_n = e^{-|\alpha|^2/2} \dfrac{\alpha^n}{\sqrt{n!}}$

$$\Rightarrow |\psi(t)\rangle = e^{-i\omega t/2} \sum_{n=0}^{\infty} e^{-|\alpha|^2/2} \frac{(\alpha e^{-i\omega t})^n}{\sqrt{n!}} |n\rangle$$

$$= e^{-i\omega t/2} \sum_{n=0}^{\infty} e^{-|\alpha(t)|^2/2} \frac{(\alpha(t))^n}{\sqrt{n!}} |n\rangle$$

where $\alpha(t) = \alpha e^{-i\omega t}$

$$\Rightarrow \boxed{\begin{array}{c} |\psi(t)\rangle = e^{-i\omega t/2} |\alpha(t)\rangle \\ \text{where } \alpha(t) = \alpha e^{-i\omega t} \end{array}}$$

At every time, the state is a coherent state with eigenvalue that evolves in time as the classical complex amplitude.

(g) At later times

$$\langle \psi(t) | \hat{x} | \psi(t) \rangle = \langle \alpha(t) | \hat{x} | \alpha(t) \rangle = x_c \, \mathrm{Re}(\alpha(t))$$
$$= x_c \, \mathrm{Re}(\alpha e^{-i\omega t})$$

$$\langle \psi(t) | \hat{p} | \psi(t) \rangle = \langle \alpha(t) | \hat{p} | \alpha(t) \rangle = p_c \, \mathrm{Im}(\alpha(t))$$
$$= p_c \, \mathrm{Im}(\alpha e^{-i\omega t})$$

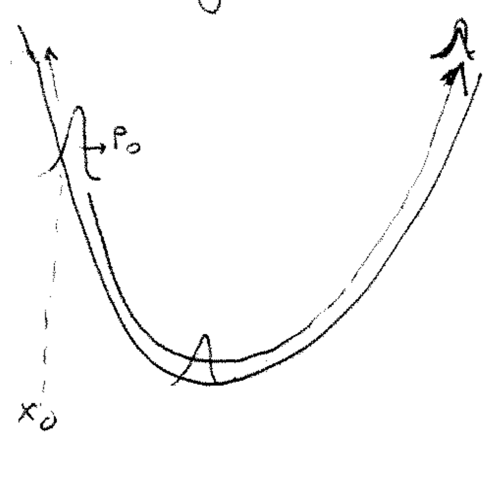
these are the classical equation of motion

(h) Wave functions:

$$\psi_{\alpha(t)}(x,t) = e^{i p(t) x / \hbar} \, u_0(x - x(t))$$

where $x(t)$ and $p(t)$ are the classical trajectories.

(i) Oscillating wave packet



Gaussian oscillating like a ~~classical~~ classical SHO

(i) $\langle \hat{N} \rangle = \langle \alpha | \hat{a}^{+} \hat{a} | \alpha \rangle = \alpha^{*} \alpha = |\alpha|^2$

$$\Delta N = \sqrt{\langle \hat{N}^2 \rangle - \langle \hat{N} \rangle^2}$$

$$\langle \hat{N}^2 \rangle = \langle \alpha | (\hat{a}^{+} \hat{a})^2 | \alpha \rangle = \langle \alpha | \hat{a}^{+} \hat{a} \hat{a}^{+} \hat{a} | \alpha \rangle$$

$$= \langle \alpha | \hat{a}^{+2} \hat{a}^{2} | \alpha \rangle + \langle \alpha | \hat{a}^{+} \underbrace{[\hat{a}, \hat{a}^{+}]}_{=1} \hat{a} | \alpha \rangle$$

$$= (\alpha^{*})^2 (\alpha)^2 + (\alpha^{*})(\alpha)$$

$$= |\alpha|^4 + |\alpha|^2$$

$$\Rightarrow \Delta N = \sqrt{|\alpha|^4 + |\alpha|^2 - |\alpha|^4} = \sqrt{|\alpha|^2} = |\alpha|$$

$$\Rightarrow \boxed{\Delta N = |\alpha|}$$

Thus $\boxed{\Delta N = \sqrt{\langle \hat{N} \rangle}}$

<u>Note:</u> $\lim_{|\alpha| \to \infty} \left(\frac{\Delta N}{\langle N \rangle} = \frac{1}{\sqrt{\langle N \rangle}} = \frac{1}{|\alpha|} \right) = 0$

The <u>fractional</u> uncertainty goes to zero as the mean amplitude $\rightarrow$ zero

(j) The probability for find the particle in the n^{th} excited states is given by

$$P_n = |\langle n|\alpha\rangle|^2 = |C_n|^2 = \left| e^{-\frac{|\alpha|^2}{2}} \frac{\alpha^n}{\sqrt{n!}} \right|^2$$

$$= e^{-|\alpha|^2} \frac{(|\alpha|^2)^n}{n!}$$

In part (i), we found $\langle n\rangle = |\alpha|^2$

$$\Rightarrow P_n = e^{-\langle n\rangle} \frac{\langle n\rangle^n}{n!}$$

This is none other than the <u>Poisson</u> distribution.

(k) The time-energy "uncertainty principle"

$\Delta E \, \Delta t \gtrsim \hbar$ (not quite a rigorous statement)

For the oscillator $E \sim n\hbar\omega$

$$\Rightarrow E t \sim n\hbar(\omega t)$$

phase of oscillator ϕ

$\Rightarrow \boxed{\Delta n \, \Delta \phi \gtrsim 1}$ This is the "number-phase" uncertainty.

A quantum oscillator with <u>definite</u> n, has uncertain <u>phase</u>