Physics 492: Quantum Mechanics II
Problem Set #4
Due: Thursday, Feb. 26, 2004

Problem 1: The Harmonic Oscillator in 3D (15 points)

Consider a particle bound harmonically in three dimensions with potential,

$$\hat{V} = \frac{1}{2} m\left(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_x^2 z^2\right).$$

(a) This potential is separable in *x*,*y*,*z*. What are the energy eigenfunctions and eigenvalues. Are there degeneracies?

(b) Let us take $\omega_x = \omega_y = \omega_z \equiv \omega$. In this case, the binding is isotropic. In this case show that the energy eigenvalues and eigenfunctions are

$$E_n = \hbar\omega\left(n + \frac{3}{2}\right), \quad n=0,1,2,...$$

$$\psi_{n_x,n_y,n_z}(x,y,z) = A_{n_x,n_y,n_z}\ \mathcal{H}_{n_x}(\sqrt{2}\bar{x})\mathcal{H}_{n_y}(\sqrt{2}\bar{y})\mathcal{H}_{n_x}(\sqrt{2}\bar{z})\ \exp\left\{-\left(\bar{x}^2 + \bar{y}^2 + \bar{z}^2\right)\right\},$$

where $n = n_x + n_y + n_z$, $\mathcal{H}_{n_x}$ is a Hermite polynomial of order n_x, and $\bar{x} = x/x_c$, etc. The degeneracy is $g_n = \frac{(n+1)(n+2)}{2}$ (extra credit if you prove it). Test this formula for the first three levels.

The degeneracy in (b) is an "essential degeneracy" due to the rotational symmetry, since for an *isotropic* SHO, the potential is a *central*, $\hat{V} = \frac{1}{2}m\omega^2 r^2$. We can now separate the Hamiltonian in spherical coordinates, with energy eigenfunctions

$$\psi_{n_r,l,m}(r,\theta,\phi) = R_{n_r,l}(r)\, Y_{l,m}(\theta,\phi),$$

where $Y_{l,m}(\theta,\phi)$ is a spherical harmonic and $R_{n_r,l}(r)$ is the radial wave function. The energy eigenvalue can be written,

$$E_{n_r,l} = \hbar\omega\left(2n_r + l + \frac{3}{2}\right).$$

(c) Consider the triply degenerate first exited state in (b) with $E = \frac{5}{2}\hbar\omega$. What the possible eigenvalues n_r, l, m?

(d) One can show that $R_{n_r,l}(r) = A\,\bar{r}^l e^{-\bar{r}^2}\mathcal{L}_{n_r}^{l+1/2}(2\bar{r}^2)$, where $\mathcal{L}_{n_r}^{l+1/2}(2\bar{r}^2)$ is an associated Laguerre polynomial. Express the state $\psi_{n_r=0,l=1,m=0}(r,\theta,\phi)$ as a superposition of eigenfunctions in part (b).

Problem 2: Angular Momentum Matrices (15 points)

(a) For spin 1/2, find the eigenvalues and eigenvector of $\hat{S}_y$.

(b) Find the matrix representations of $\hat{S}_x$, $\hat{S}_y$, and $\hat{S}_z$ in this basis – comment.

(c) For each of the eigenvectors of $\hat{S}_y$, what is the probability of finding spin-up or spin-down along the z-direction. Along the x-direction?

(d) For j=1, find the eigenvalues and eigenvectors of $\hat{J}_x$.

(e) Find the matrix representations of $\hat{J}_x$, $\hat{J}_y$, and $\hat{J}_z$ in this basis.

(f) For each of the eigenvectors of $\hat{J}_x$, what are the probabilities of finding different projections of angular momentum along the z-axis.