Physics 492 - Quantum II

Problem Set #4 - Solutions

Problem 1: The Harmonic Oscillator in 3D

$$\hat{V} = \frac{1}{2} m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)$$

This potential is separable in Cartesian coordinates

$$\hat{V} = \hat{V}_x + \hat{V}_y + \hat{V}_z$$

Since $\frac{\hat{\vec{p}}^2}{2m} = \frac{\hat{p}_x^2}{2m} + \frac{\hat{p}_y^2}{2m} + \frac{\hat{p}_z^2}{2m}$

the Hamiltonian is separable $\hat{H} = \hat{H}_x + \hat{H}_y + \hat{H}_z$

Where $\hat{H}_x = \frac{\hat{p}_x^2}{2m} + \frac{1}{2} m\omega_x^2 x^2$, etc. for y, z

There thus exist eigenstates states of $\hat{H}$ which are products of the eigenstates of $\hat{H}_x, \hat{H}_y, \hat{H}_z$

$$\psi_{n_x, n_y, n_z}(x, y, z) = u_{n_x}(x)\, u_{n_y}(y)\, u_{n_z}(z)$$

Where $\hat{H}_{n_x} u_{n_x}(x) = E_{n_x} u_{n_x}(x)$ etc

with $E_{n_x} = \hbar\omega(n_x + \frac{1}{2})$

and $u_{n_x}(x) = A_{n_x} \mathcal{H}_{n_x}(\sqrt{2}\,\bar{x})\, e^{-\bar{x}^2}$ $\quad \bar{x} = x/x_c$

Thus the total energy eigenvalue is

$$\boxed{\begin{aligned} E_{n_x,n_y,n_z} &= \hbar(\omega_x n_x + \omega_y n_y + \omega_z n_z) + \frac{3}{2} \\ \psi_{n_x,n_y,n_z} &= A_{n_x,n_y,n_z} \mathcal{H}_{n_x}(\sqrt{2}x)\, \mathcal{H}_{n_y}(\sqrt{2}y)\, \mathcal{H}_{n_z}(\sqrt{2}z) \\ & \qquad \exp\{-\bar{x}^2 - \bar{y}^2 - \bar{z}^2\} \end{aligned}}$$

(b) Isotropic case: $\omega_x = \omega_y = \omega_z \equiv \omega$

$$\Rightarrow x_c = y_c = z_c \equiv r_c = \sqrt{\frac{2\hbar}{m\omega}}$$

$$\Rightarrow E = \hbar\omega(n_x + n_y + n_z + 3/2) = \hbar\omega(n + 3/2)$$

Note: energy only depends on sum of integers $n_x + n_y + n_z \equiv n$

$\Rightarrow$ degeneracy g_n

To calculate this degeneracy, fix one quantum number, say n_z

$$\Rightarrow n_x + n_y = n - n_z \qquad \Rightarrow g_n^{3D} = \sum_{n_z=0}^{n} g^{2D}_{n-n_z}$$

In 2D (say $n_z = 0$) the degeneracy is $n_{2D} + 1$

$$\Rightarrow g_n^{3D} = \sum_{n_z=0}^{n} (n - n_z + 1) = (n+1)^2 - \sum_{n_z=0}^{n} n_z = (n+1)^2 - \frac{n(n+1)}{2}$$

$$\Rightarrow \boxed{g_n = \frac{(n+1)(n+2)}{2}}$$

In the isotropic case

$$\hat{V} = \frac{1}{2}m\omega^2(\hat{x}^2+\hat{y}^2+\hat{z}^2) = \frac{1}{2}m\omega^2\hat{r}^2$$

$$(\text{where } r^2 = |\vec{r}|^2 = x^2+y^2+z^2)$$

So this is a <u>central potential</u> $V(r)$

$\Rightarrow$ There are energy eigenfunctions specified as the simultaneous eigenfunctions of

$$\{\hat{H}, \hat{L}^2, \hat{L}_z\} \leftarrow \text{complete set of mutually commuting ops.}$$

$$\psi_{n_r,l,m}(r,\theta,\phi,) = \underbrace{R_{n_r,l}(r)}_{\substack{\text{radial}\\ \text{wave function}\\ n_r=0,1,\ldots}} \underbrace{Y_{l,m}(\theta,\phi)}_{\substack{\text{Spherical}\\ \text{Harmonic}}} \quad \begin{matrix} l=0,1,\ldots \\ -l\le m_l \le l\end{matrix}$$

Where $E_{n_r,l} = \hbar\omega(\underbrace{2n_r+l}_{n = 0,1,2,\ldots}+3/2)$

(c) First excited state $n=1$ $\quad E = \hbar\omega(1+3/2) = \frac{5}{2}\hbar\omega$

$\Rightarrow 2n_r + l = 1 \Rightarrow l = 1$

$\Rightarrow m_l = -1, 0, +1$

$\Rightarrow$ Degeneracy of 3, as expected.

(d) The radial wave function can be shown to be

$$R_{n_r, l}(r) = A \bar{r}^{\,l} e^{-\bar{r}^2} \mathcal{L}_{n_r}^{l+\frac{1}{2}}(2\bar{r}^2)$$

Case $n_r = 0$ $l = 1$ $E = \frac{5}{2}\hbar\omega$ (triply degenerate first excited state)

$$\psi_{n_r=0, l=1, m}(r, \theta, \phi) = A e^{-\bar{r}^2} \bar{r} Y_{l=1, m}(\theta, \phi)$$

Recall $Y_{l=1, m=\pm 1} = C_{\pm 1} \dfrac{x + iy}{r}$

$$Y_{l=1, m=0} = C_0 \frac{z}{r}$$

$$\Rightarrow \psi_{n_r=0, l=1, m=0}(r, \theta, \phi) = \psi_{n_x=0, n_y=0, n_z=1}(x, y, z)$$

Whereas

$$\psi_{n_r=0, l=1, m=\pm 1}(r, \theta, \phi) = \frac{\psi_{n_x=1, n_y=0, n_z=0} \pm i\, \psi_{n_x=0\, n_y=1\, n_z=0}}{\sqrt{2}}$$

Note: In general $[\hat{L}_z, \hat{H}_{x,y,z}] \neq 0$ so

the separation of variable in Cartesian + spherical coordinates do not yield the same eigenfunctions

Problem 2: Angular Momentum Matrices

(a) Spin-1/2, we found in the standard basis $\{|\uparrow_z\rangle, |\downarrow_z\rangle\}$

$$\hat{S}_y \doteq \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \frac{\hbar}{2}\hat{\sigma}_y$$

We know the eigenvalues are $\pm\frac{\hbar}{2}$ (same as $\hat{S}_z$)
To prove this

$$\det[\hat{\sigma}_y - \lambda \mathbb{1}] = \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1 \quad ✓$$

Eigenvectors: $\frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} c_\uparrow^\pm \\ c_\downarrow^\pm \end{bmatrix} = \lambda \begin{bmatrix} c_\uparrow^\pm \\ c_\downarrow^\pm \end{bmatrix} = \pm\frac{\hbar}{2}\begin{bmatrix} c_\uparrow^\pm \\ c_\downarrow^\pm \end{bmatrix}$

$$\Rightarrow -i\frac{\hbar}{2} c_\downarrow^\pm = \pm\frac{\hbar}{2} c_\uparrow^\pm \Rightarrow c_\downarrow^\pm = \pm i\, c_\uparrow^\pm$$

$\Rightarrow$ Unnormalized $|\uparrow_y\rangle \doteq \begin{bmatrix} 1 \\ i \end{bmatrix}$ $\quad |\downarrow_y\rangle \doteq \begin{bmatrix} 1 \\ -i \end{bmatrix}$

$\Rightarrow$ Normalized

$$\boxed{\begin{aligned} |\uparrow_y\rangle &= \frac{|\uparrow_z\rangle + i\,|\downarrow_z\rangle}{\sqrt{2}} \\ |\downarrow_y\rangle &= \frac{|\uparrow_z\rangle - i\,|\downarrow_z\rangle}{\sqrt{2}} \end{aligned}}$$

(b) We seek representations of operators in the basis $\{ |\uparrow_y\rangle, |\downarrow_y\rangle \}$

Recall $$\hat{\sigma}_n = \hat{\mathbb{1}}\, \hat{\sigma}_n\, \hat{\mathbb{1}} = \sum_{m_z, m_z' = -1}^{1} |m_z\rangle \underbrace{\langle m_z | \hat{\sigma}_n | m_z' \rangle}_{\text{Matrix elements in standard basis}} \langle m_z' |$$

$\Rightarrow$ Matrix elements in new basis

$$\langle m_y | \hat{\sigma}_n | m_y' \rangle = \sum_{m_z, m_z'} \langle m_y | m_z \rangle \langle m_z | \hat{\sigma}_n | m_z' \rangle \langle m_z' | m_y' \rangle$$

Let $U^*_{m_y, m_z} = \langle m_y | m_z \rangle$

$$\Rightarrow \quad \hat{U} \doteq \begin{bmatrix} \langle \uparrow_y | \uparrow_z \rangle & \langle \uparrow_y | \downarrow_z \rangle \\ \langle \downarrow_y | \uparrow_z \rangle & \langle \downarrow_y | \downarrow_z \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}, \qquad \hat{U}^\dagger = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$$

$\therefore$ In basis $\{ |\uparrow_y\rangle, |\downarrow_y\rangle \}$ we make a similarity transform

$$\hat{\sigma}_x \doteq \frac{1}{2} \underset{\hat{U}}{\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\hat{\sigma}_x \text{ in standard basis}} \underset{\hat{U}^\dagger}{\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}}$$

$$\Rightarrow \boxed{\begin{aligned}
\hat{\sigma}_x &\doteq \frac{1}{2}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\
\hat{\sigma}_y &\doteq \frac{1}{2}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
\hat{\sigma}_z &\doteq \frac{1}{2}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\end{aligned}}$$

Thus, in this new basis the Pauli operators have a "permuted" matrix representation

$y \rightarrow z \qquad z \rightarrow x \qquad x \rightarrow y$

(c) We found $|\uparrow_y\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle + i\,|\downarrow_z\rangle\right)$

$$|\downarrow_y\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle - i\,|\downarrow_z\rangle\right)$$

$$|\uparrow_x\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle + |\downarrow_z\rangle\right)$$

$$|\downarrow_x\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle - |\downarrow_z\rangle\right)$$

For $|\uparrow_y\rangle$

$$P_{\uparrow_z} = |\langle\uparrow_z|\uparrow_y\rangle|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

$$P_{\uparrow_x} = |\langle\uparrow_x|\uparrow_y\rangle|^2 = \left|\frac{1+i}{2}\right|^2 = \frac{1}{2}$$

$$P_{\downarrow_z} = |\langle\downarrow_z|\uparrow_y\rangle|^2 = \left|\frac{i}{\sqrt{2}}\right|^2 = 1/2$$

$$P_{\downarrow_x} = |\langle\downarrow_x|\uparrow_y\rangle|^2 = \left|\frac{1-i}{2}\right|^2 = 1/2$$

For $|\downarrow_y\rangle =$

$$P_{\uparrow_z} = |\langle\uparrow_z|\downarrow_y\rangle|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

$$P_{\uparrow_x} = |\langle\uparrow_x|\downarrow_y\rangle|^2 = \left|\frac{1-i}{2}\right|^2 = \frac{1}{2}$$

$$P_{\downarrow_z} = |\langle\downarrow_z|\downarrow_y\rangle|^2 = \left|\frac{-i}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

$$P_{\downarrow_x} = |\langle\downarrow_x|\downarrow_y\rangle|^2 = \left|\frac{1+i}{2}\right|^2 = \frac{1}{2}$$

Thus $\left(\begin{array}{l}|\uparrow_y\rangle \\ \text{and } |\downarrow_y\rangle\end{array}\right)$ is 50-50 superposition of $|\uparrow_z\rangle, |\downarrow_z\rangle$
or $|\uparrow_x\rangle, |\downarrow_x\rangle$

The phase relationship of this superposition determines which state

$$|\uparrow_y\rangle = \frac{1}{\sqrt{2}}|\uparrow_z\rangle + \frac{i}{\sqrt{2}}|\downarrow_z\rangle = e^{i\pi/4}\left(\frac{1}{\sqrt{2}}|\uparrow_x\rangle - \frac{i}{\sqrt{2}}|\downarrow_x\rangle\right)$$

$$|\downarrow_y\rangle = \frac{1}{\sqrt{2}}|\uparrow_z\rangle - \frac{i}{\sqrt{2}}|\downarrow_z\rangle = e^{-i\pi/4}\left(\frac{1}{\sqrt{2}}|\uparrow_x\rangle + \frac{i}{\sqrt{2}}|\downarrow_x\rangle\right)$$

(d) We seek the eigenvalues of: $\hat{J}_x = \hbar\hat{j}_x$ for $j = 1$

We found in class $\hat{j}_x \doteq \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

in the "standard basis" $\{|j=1, m=1\rangle, |j=1, m=0\rangle, |j=1, m=-1\rangle\}$

$$\hat{j}_x \underbrace{|j=1, m_x\rangle}_{\text{eigenvector}} = \underset{\substack{\uparrow \\ \text{eigenvalue}}}{m_x} |j=1, m_x\rangle$$

$$\Rightarrow \det\left[\hat{j}_x - m_x \mathbb{1}\right] = 0$$ (Characteristic eqn)

$$\det \begin{bmatrix} -m_x & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -m_x & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -m_x \end{bmatrix} = -m_x\left(m_x^2 - \frac{1}{2}\right) - \frac{1}{\sqrt{2}}\left(-\frac{m_x}{\sqrt{2}}\right) = 0$$

$$\Rightarrow m_x\left(m_x^2 - 1\right) = 0$$

$\Rightarrow$ Three roots : $\boxed{m_x = 0, \quad m_x = 1, \quad m_x = -1}$

This are, of course, the <u>same eigenvalues</u> as $\hat{j}_z$ since the direction in space is arbitrary.

Eigenvectors:

$$|j=1, m_x=1\rangle \doteq \begin{bmatrix} c_1 \\ c_0 \\ c_{-1} \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_0 \\ c_{-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow c_1 = \frac{c_0}{\sqrt{2}} \qquad c_1 = \frac{c_0}{\sqrt{2}} \Rightarrow |j=1, m_x=1\rangle \doteq \frac{c_0}{\sqrt{2}} \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$$

Normalization $\left|\langle j=1, m_x=1 | j=1, m=1\rangle\right|^2 = |c_0|^2 = 1$

$$\Rightarrow c_0 = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \boxed{|j=1, m_x=1\rangle \doteq \frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} = \frac{|m_z=1\rangle}{2} + \frac{1}{\sqrt{2}}|m_z=0\rangle + \frac{|m_z=-1\rangle}{\sqrt{2}}}$$

$$|j=1, m_x=0\rangle = \begin{bmatrix} d_1 \\ d_0 \\ d_{-1} \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_0 \\ d_{-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow d_0 = 0 \quad d_1 = -d_{-1} \quad \Rightarrow \text{Normalized } \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\therefore \boxed{|j=1, m_x=0\rangle = \frac{1}{\sqrt{2}} \left(|m_z=1\rangle - |m_z=-1\rangle \right)}$$

$$|j=1, m_x=-1\rangle = \begin{bmatrix} e_1 \\ e_0 \\ e_{-1} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_0 \\ e_{-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow e_1 = -\frac{e_0}{\sqrt{2}} = e_{-1} \Rightarrow |j=1, m_x=-1\rangle \doteq \frac{e_0}{\sqrt{2}} \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$$

$$\Rightarrow \boxed{|j=1, m_x=-1\rangle \doteq \frac{1}{2} \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix} = \frac{1}{2}|m_z=1\rangle - \frac{1}{\sqrt{2}}|m_z=0\rangle + \frac{1}{2}|m_z=-1}$$

Note: You can check for yourself that these eigenvectors are orthogonal as they must be for a Hermitian matrix with nondegenerate eigenvalues

(e) We can find matrix representations in this basis by performing a similarity transformation

$$\underset{\substack{\nearrow \\ \text{rep in} \\ \text{new basis}}}{\tilde{J}_i} = U^{\dagger} \underset{\substack{\nwarrow \\ \text{standard} \\ \text{basis rep}}}{J_i} U \quad , \quad U_{m_z, m_x} = \langle m_z | m_x \rangle$$

$$\Rightarrow \quad U = \begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ 1/\sqrt{2} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{bmatrix}$$

$\Rightarrow$ In basis $\{ |m_x=1\rangle, |m_x=0\rangle, |m_x=-1\rangle \}$

$$\hat{J}_x \doteq \frac{1}{4\sqrt{2}} \begin{bmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{bmatrix} = \begin{array}{c} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{array} \right] \end{array} \begin{array}{l} \langle m_x = 1 \\ \langle m_x = 0 \\ \langle m_x = -1 \end{array}$$

$$\hat{J}_y \doteq \frac{i}{4\sqrt{2}} \begin{bmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{bmatrix} = \frac{i}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\hat{J}_z = \frac{1}{4} \begin{bmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\text{Thus} \quad \left. \begin{aligned} \hat{J}_x \Big|_{m_x \text{ representation}} &= \hat{J}_z \Big|_{m_z \text{ representation}} \\ \hat{J}_z \Big|_{m_x \text{ rep}} &= \hat{J}_x \Big|_{mx \text{ rep.}} \\ \hat{J}_y \Big|_{mx \text{ rep}} &= - \hat{J}_y \Big|_{m_z \text{ rep.}} \end{aligned} \right\}$$

(f) Each eigenvector of $\hat{J}_x$

$$|m_x=1\rangle \doteq \begin{array}{cl} \left[\begin{matrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{matrix}\right] & \begin{matrix} \langle m_z=1| \\ \langle m_z=0| \\ \langle m_z=-1| \end{matrix} \end{array} \Rightarrow \begin{aligned} P_{m_z=1} &= \frac{1}{4} \\ -1 &= \frac{1}{2} \\ 0 &= \frac{1}{4} \end{aligned}$$

$$|m_x=0\rangle \doteq \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \Rightarrow \begin{aligned} P_{m_z=1} = P_{m_z=-1} &= \frac{1}{2} \\ P_{m_z=0} &= 0 \end{aligned}$$

$$|m_x=-1\rangle \doteq \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix} \Rightarrow \begin{aligned} P_{m_z=1} = P_{m_z=-1} &= \frac{1}{4} \\ P_{m_z=0} &= \frac{1}{2} \end{aligned}$$