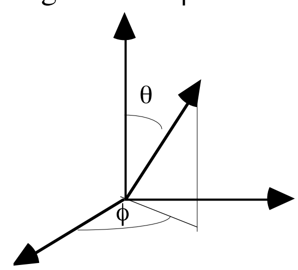
Physics 492: Quantum Mechanics II
Problem Set #5
Due: Wed, March 3, 2004

Problem 1: Spin-1/2 along an arbitrary direction (10 points)
Given a unit vector $\vec{\mathbf{e}}_n$, defined by angles θ and ϕ with respect to the polar axis z,



(a) Show that the state $|\uparrow_n\rangle = \cos(\theta/2)|\uparrow_z\rangle + e^{i\phi}\sin(\theta/2)|\downarrow_z\rangle$, where $\{|\uparrow_z\rangle, |\downarrow_z\rangle\}$ is the standard basis, is an eigenstate of $\hat{\sigma}_n \equiv \vec{\mathbf{e}}_n \cdot \hat{\vec{\sigma}}$ with eigenvalue 1, $\hat{\sigma}_n|\uparrow_n\rangle = |\uparrow_n\rangle$. Thus we interpret $|\uparrow_n\rangle$ as "spin-up along the direction $\vec{\mathbf{e}}_n$".

(b) Show that $|\uparrow_{-n}\rangle = |\downarrow_n\rangle$. Check for $\vec{\mathbf{e}}_n = \vec{\mathbf{e}}_z$.

(c) Use (a) and (b) to express $\{|\uparrow_x\rangle, |\downarrow_x\rangle, |\uparrow_y\rangle, |\downarrow_y\rangle\}$ in the standard basis.

(d) **Extra Credit:** Show that the inner product between any two pure states is, $|\langle\uparrow_n|\uparrow_{n'}\rangle| = \cos(\Theta/2)$, where Θ is the angle between the directions $\vec{\mathbf{e}}_n$ and $\vec{\mathbf{e}}_{n'}$ in three dimensional space. Is this consistent with the statement "spin-up and spin-down are orthogonal states"?

Problem 2: The Rotation Operator for spin-1/2 (10 points)

We have learned that the operator $R_n(\Theta) = \exp\{-i\Theta(\vec{\mathbf{e}}_n \cdot \hat{\mathbf{J}})/\hbar\}$ is a "rotation operator", which rotates a vector about the axis $\vec{\mathbf{e}}_n$ by an angle Θ. For the case of spin 1/2,

$$\hat{\mathbf{J}} = \hat{\mathbf{S}} = \frac{\hbar}{2}\hat{\vec{\sigma}} \Rightarrow R_n(\Theta) = \exp\{-i\Theta\hat{\sigma}_n/2\}.$$

(a) Show that for spin 1/2, $R_n(\Theta) = \cos\left(\frac{\Theta}{2}\right)\hat{1} - i\sin\left(\frac{\Theta}{2}\right)\hat{\sigma}_n$. (Hint: Expand exponential)

(b) Show: $R_n(\Theta = 2\pi) = -\hat{1}$ -- Comment.

(c) Consider a series of rotations: Rotate about the y-axis by θ followed by a rotation about the z-axis by ϕ. Convince yourself that this takes the unit vector along $\vec{\mathbf{e}}_z$ to $\vec{\mathbf{e}}_n$. Show that up to an overall phase,

$$\left|\uparrow_n\right\rangle = R_z(\phi) R_y(\theta) \left|\uparrow_z\right\rangle .$$