Problem 1: Spin-1/2 along an arbitrary direction (10 points)
Given a unit vector $\vec{e}_n$, defined by angles $\theta$ and $\phi$ with respect to the polar axis $z$,

(a) Show that the state $|\uparrow_n\rangle = \cos(\theta/2)|\uparrow_z\rangle + e^{i\phi}\sin(\theta/2)|\downarrow_z\rangle$, where $\{|\uparrow_z\rangle,|\downarrow_z\rangle\}$ is the standard basis, is an eigenstate of $\hat{\sigma}_n \equiv \vec{e}_n \cdot \hat{\sigma}$ with eigenvalue 1, $\hat{\sigma}_n |\uparrow_n\rangle = |\uparrow_n\rangle$. Thus we interpret $|\uparrow_n\rangle$ as “spin-up along the direction $\vec{e}_n$”.

(b) Show that $|\uparrow_{-n}\rangle = |\downarrow_n\rangle$. Check for $\vec{e}_n = \hat{e}_z$.

(c) Use (a) and (b) to express $|\uparrow_x\rangle, |\downarrow_x\rangle, |\uparrow_y\rangle, |\downarrow_y\rangle$ in the standard basis.

(d) **Extra Credit:** Show that the inner product between any two pure states is, $\langle\uparrow_n|\uparrow_{n'}\rangle = \cos(\Theta/2)$, where $\Theta$ is the angle between the directions $\vec{e}_n$ and $\vec{e}_{n'}$ in three-dimensional space. Is this consistent with the statement “spin-up and spin-down are orthogonal states”?

Problem 2: The Rotation Operator for spin-1/2 (10 points)
We have learned that the operator $R_n(\Theta) = \exp\{-i\Theta(\hat{e}_n \cdot \hat{J})/\hbar\}$ is a “rotation operator”, which rotates a vector about the axis $\hat{e}_n$ by an angle $\Theta$. For the case of spin 1/2,

$$\hat{J} = \hat{S} = \frac{\hbar}{2} \hat{\sigma} \Rightarrow R_n(\Theta) = \exp\{-i\Theta \hat{\sigma}_n/2\}.$$ 

(a) Show that for spin 1/2, $R_n(\Theta) = \cos(\Theta/2)\hat{J} - i\sin(\Theta/2)\hat{\sigma}_n$. (Hint: Expand exponential)

(b) Show: $R_n(\Theta = 2\pi) = -\hat{1}$ -- Comment.

(c) Consider a series of rotations: Rotate about the $y$-axis by $\theta$ followed by a rotation about the $z$-axis by $\phi$. Convince yourself that this takes the unit vector along $\hat{e}_z$ to $\hat{e}_n$. Show that up to an overall phase,
\[ | \uparrow, n \rangle = R_z(\phi) R_y(\theta) | \uparrow, z \rangle. \]