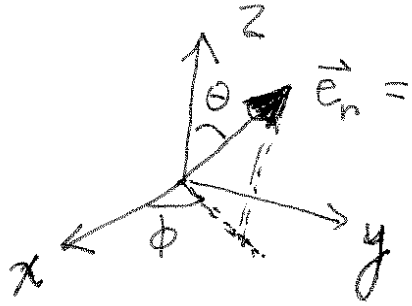
Physics 492 - Quantum II

Problem Set #5 - Solutions

Problem 1: Spin-1/2 along an arbitrary direction



$$\vec{e}_n = \sin\theta\,(\cos\phi\,\vec{e}_x + \sin\phi\,\vec{e}_y) + \cos\theta\,\vec{e}_z$$

Unit vector defined by polar angles

Define $\quad |\uparrow_n\rangle = \cos\left(\frac{\theta}{2}\right)|\uparrow_z\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|\downarrow_z\rangle$

$$\hat{\sigma}_n \equiv \vec{e}_n\cdot\hat{\vec{\sigma}} = \sin\theta\cos\phi\,\hat{\sigma}_x + \sin\theta\sin\phi\,\hat{\sigma}_y + \cos\theta\,\hat{\sigma}_z$$

Using matrix representation in the standard bases,

$$|\uparrow_n\rangle \doteq \begin{bmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{bmatrix}$$

$$\hat{\sigma}_n \doteq \begin{bmatrix} \cos\theta & \sin\theta\,(\cos\phi - i\sin\phi) \\ \sin\theta\,(\cos\phi + i\sin\phi) & -\cos\theta \end{bmatrix} =$$

$$= \begin{bmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{bmatrix}$$

(a)

$$\Rightarrow \hat{\sigma}_n |\uparrow_n\rangle = \begin{bmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\frac{\theta}{2} + \sin\theta\sin\frac{\theta}{2} \\ e^{i\phi}\left(\sin\theta\cos\frac{\theta}{2} - \cos\theta\sin\frac{\theta}{2}\right) \end{bmatrix} = \begin{bmatrix} \cos\left(\theta - \frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\theta - \frac{\theta}{2}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{bmatrix} \doteq |\uparrow_n\rangle \quad \checkmark$$

(b) $|\uparrow_{-n}\rangle$ = eigenstate with $\begin{pmatrix} \theta \to \pi - \theta \\ \phi \to -\phi \end{pmatrix}$ (i.e. $\vec{e}_n \to -\vec{e}_n$)

$$\Rightarrow |\uparrow_{-n}\rangle = \cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right)|\uparrow_z\rangle - e^{i\phi}\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)|\downarrow_z\rangle$$

$$\boxed{|\uparrow_{-n}\rangle = \sin\frac{\theta}{2}|\uparrow_z\rangle - e^{i\phi}\cos\frac{\theta}{2}|\downarrow_z\rangle}$$

$$\hat{\sigma}_n |\uparrow_{-n}\rangle \doteq \begin{bmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} \sin\frac{\theta}{2} \\ -e^{i\phi}\cos\frac{\theta}{2} \end{bmatrix} = \begin{bmatrix} -\sin\left(\theta - \frac{\theta}{2}\right) \\ e^{i\phi}\cos\left(\theta - \frac{\theta}{2}\right) \end{bmatrix}$$

$$= \begin{bmatrix} -\sin\frac{\theta}{2} \\ e^{i\phi}\cos\frac{\theta}{2} \end{bmatrix} \doteq -|\uparrow_{-n}\rangle$$

(eigenstate of $\hat{\sigma}_n$ with eigenvalue -1)

$$\Rightarrow |\uparrow_{-n}\rangle = |\downarrow_n\rangle$$

Check $\vec{e}_z$: $\theta = 0 \quad \phi = 0$

$$\Rightarrow |\uparrow_{\vec{e}_z}\rangle = |\uparrow_z\rangle$$

$-\vec{e}_z$: $\theta = \pi \quad \phi = 0$

$$\Rightarrow |\uparrow_{-\vec{e}_z}\rangle = |\downarrow_{\hat{e}_z}\rangle$$

(c) Other axes: $\vec{e}_x$: $\theta = \frac{\pi}{2}$, $\phi = 0$
$-\vec{e}_x$: $\theta = \frac{\pi}{2}$, $\phi = \pi$

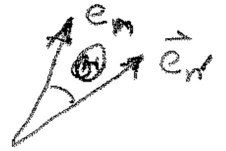
$$\Rightarrow |\uparrow_x\rangle = \cos\left(\frac{\pi}{4}\right)|\uparrow_z\rangle + \sin\left(\frac{\pi}{4}\right)|\downarrow_z\rangle$$
$$= \frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle + |\downarrow_z\rangle\right)$$

$$|\downarrow_x\rangle = \cos\left(\frac{\pi}{4}\right)|\uparrow_z\rangle + e^{i\pi}\sin\left(\frac{\pi}{4}\right)|\downarrow_z\rangle$$
$$= \frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle - |\downarrow_z\rangle\right)$$

y-axis: $\vec{e}_y : \theta = \frac{\pi}{2}$, $\phi = \frac{\pi}{2}$
$-\vec{e}_y : \theta = \frac{\pi}{2}$, $\phi = 3\pi/2$

$$\Rightarrow |\uparrow_y\rangle = \cos\left(\frac{\pi}{4}\right)|\uparrow_z\rangle + e^{i\frac{\pi}{2}}\sin\left(\frac{\pi}{4}\right)|\downarrow_z\rangle = \frac{|\uparrow_z\rangle + i|\downarrow_z\rangle}{\sqrt{2}}$$
$$|\downarrow_y\rangle = \cos\left(\frac{\pi}{4}\right)|\uparrow_z\rangle + e^{i3\pi/2}\sin\left(\frac{\pi}{4}\right)|\downarrow_z\rangle = \frac{|\uparrow_z\rangle - i|\downarrow_z\rangle}{\sqrt{2}}$$

(d) Consider two directions in space



Θ is the angle between these

$$\cos\Theta = \vec{e}_n \cdot \vec{e}_{n'} = \cos\theta\cos\theta' + \sin\theta\sin\theta'(\cos\phi\cos\phi' + \sin\phi\sin\phi')$$

$$\Rightarrow \cos\Theta = \cos\theta\cos\theta' + \cos(\phi-\phi')\sin\theta\sin\theta'$$

(where θ, ϕ and θ', ϕ' define the directions $\vec{e}_n$ and $\hat{e}_{n'}$ respectively)

Now $|\uparrow_n\rangle = \cos\frac{\theta}{2}|\uparrow_z\rangle + e^{i\phi}\sin\frac{\theta}{2}|\downarrow_z\rangle$

$|\uparrow_{n'}\rangle = \cos\frac{\theta'}{2}|\uparrow_z\rangle \quad e^{i\phi'}\sin\frac{\theta}{2}|\downarrow_z\rangle$

$$\Rightarrow \langle\uparrow_n|\uparrow_{n'}\rangle = \cos\frac{\theta}{2}\cos\frac{\theta'}{2} + e^{i(\phi-\phi')}\sin\frac{\theta}{2}\sin\frac{\theta'}{2}$$

$$|\langle\uparrow_n|\uparrow_{n'}\rangle|^2 = \cos^2\frac{\theta}{2}\cos^2\frac{\theta'}{2} + \sin^2\frac{\theta}{2}\sin^2\frac{\theta'}{2} + 2\,\mathrm{Re}\left(e^{i(\phi-\phi')}\cos\frac{\theta}{2}\cos\frac{\theta'}{2}\sin\frac{\theta}{2}\sin\frac{\theta'}{2}\right)$$

Aside: Use trig identities $\left(\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2} \quad \sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}\right.$

$\left. 2\sin\theta\cos\theta = \sin 2\theta \right)$

$$\Rightarrow |\langle\uparrow_n|\uparrow_{n'}\rangle|^2 = \left(\frac{1+\cos\theta}{2}\right)\left(\frac{1+\cos\theta'}{2}\right) + \left(\frac{1-\cos\theta}{2}\right)\left(\frac{1-\cos\theta'}{2}\right) + \frac{1}{2}\sin\theta\sin\theta'\cos(\phi-\phi')$$

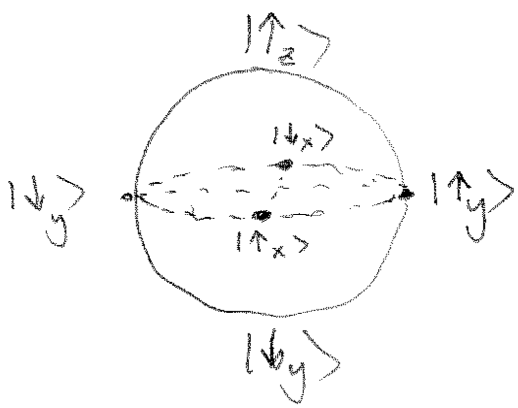
$$\Rightarrow |\langle \uparrow_n | \uparrow_{n'} \rangle|^2 = \frac{1}{2}\left(1 + \cos\theta\cos\theta' + \sin\theta\sin\theta' \cos(\phi - \phi')\right)$$

$$= \frac{1}{2}\left(1 + \cos\Theta\right)$$

$$|\langle \uparrow_n | \uparrow_{n'} \rangle|^2 = \cos^2\left(\frac{\Theta}{2}\right)$$

$$\Rightarrow \boxed{|\langle \uparrow_n | \uparrow_{n'} \rangle| = \cos\left(\frac{\Theta}{2}\right)}$$

The inner product between two spin states in Hilbert space is related to <u>half the ~~a~~ angle</u> between the directions in physical space



The sphere (known as the "Bloch Sphere") represents Hilbert space for spin $\frac{1}{2}$

<u>Note:</u> <u>Antipodal points</u> are orthogonal in Hilbert space $\Theta = \pi$

Problem 2: Rotation Operator for spin-1/2

For spin-1/2, rotation operator by Θ about $\vec{e}_n$

$$R_n(\Theta) = e^{-i\frac{\Theta}{2}\hat{\sigma}_n}$$ (as described in problem)

(a) $$R_n(\Theta) = \sum_{m=0}^{\infty} \left(-i\frac{\Theta}{2}\right)^m \frac{(\hat{\sigma}_n)^m}{m!}$$

Aside: $\hat{\sigma}_n^2 = \hat{\mathbb{1}}$ (Pauli operators square to unity)

Thus, we separate the sum into even and odd terms

$$(\hat{\sigma}_n)^m = \begin{cases} \hat{\mathbb{1}} & m \text{ even} \\ \hat{\sigma}_n & m \text{ odd} \end{cases}$$

Moreover, $$(-i)^m = \begin{cases} (-1)^{m/2} & m = 0, 2, \ldots \\ -i\,(-1)^{\frac{m-1}{2}} & m = 1, 3, 5 \ldots \end{cases}$$

$$\Rightarrow \hat{R}_n(\Theta) = \left(\sum_{m\text{ even}=0}^{\infty} (-1)^{m/2}\left(\frac{\Theta}{2}\right)^m \frac{1}{m!}\right)\hat{\mathbb{1}}$$

$$-i\left(\sum_{m\text{ odd}=1}^{\infty} (-1)^{\frac{m-1}{2}} \left(\frac{\Theta}{2}\right)^m \frac{1}{m!}\right)\hat{\sigma}_n$$

$$\Rightarrow \boxed{\hat{R}_n(\Theta) = \cos\frac{\Theta}{2}\,\hat{\mathbb{1}} - i\sin\frac{\Theta}{2}\,\hat{\sigma}_n}$$

(b) $\hat{R}_n(\Theta = 2\pi) = \cos\pi\, \hat{\mathbb{1}} - i \sin\pi\, \hat{\sigma}_n$

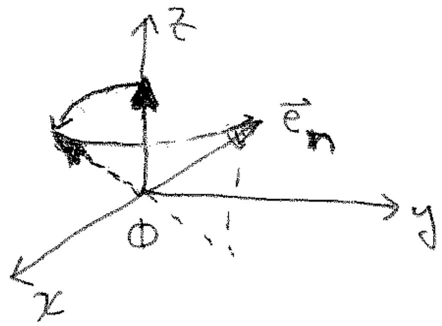
$$\Rightarrow \boxed{\hat{R}_n(\Theta = 2\pi) = -\hat{\mathbb{1}}}$$

Thus $\hat{R}_n(\Theta = 2\pi)|\psi\rangle = -|\psi\rangle$

For and $|\psi\rangle$ describing a spin $\frac{1}{2}$

This is the famous phase associated with spinors, that differentiates it from orbital angular momentum.

(c)



We obtain a vector in the (θ, ϕ) direction by first rotating about y by θ and the about z by ϕ

$$\hat{R}_y(\theta) = \cos\frac{\theta}{2}\,\hat{\mathbb{1}} - i \sin\frac{\theta}{2}\,\hat{\sigma}_y$$

$$\doteq \begin{array}{c} \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} \end{array} \begin{array}{c} \langle\uparrow_z| \\ \langle\downarrow_z| \end{array}$$

(matrix in standard basis)

$$\Rightarrow \hat{R}_y(\theta)|\uparrow_z\rangle \doteq \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{bmatrix}$$

Now follow this by a rotation about z by ϕ

$$\hat{R}_z(\phi) = \cos\frac{\phi}{2}\,\hat{1} - i\sin\frac{\phi}{2}\,\hat{\sigma}_z$$

$$\doteq \begin{bmatrix} \cos\frac{\phi}{2} - i\sin\frac{\phi}{2} & 0 \\ 0 & \cos\frac{\phi}{2} + i\sin\frac{\phi}{2} \end{bmatrix} = \begin{bmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{bmatrix}$$

$$\Rightarrow \hat{R}_z(\phi)\left(\hat{R}_y(\theta)\,|\uparrow_z\rangle\right) \doteq \begin{bmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{bmatrix}\begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{bmatrix}$$

$$\doteq \begin{bmatrix} e^{-i\phi/2}\cos\frac{\theta}{2} \\ e^{i\phi/2}\sin\frac{\theta}{2} \end{bmatrix}$$

$$= e^{-i\phi/2}\cos\frac{\theta}{2}\,|\uparrow_z\rangle + e^{i\phi/2}\sin\frac{\theta}{2}\,|\downarrow_z\rangle$$

$$= \cancel{e^{-i\phi/2}}\left(\cos\frac{\theta}{2}\,|\uparrow_z\rangle + e^{i\phi}\sin\frac{\theta}{2}\,|\downarrow_z\rangle\right)$$

neglect

$$\Rightarrow \hat{R}_z(\phi)\,\hat{R}_y(\theta)\,|\uparrow_z\rangle \cong |\uparrow_n\rangle$$