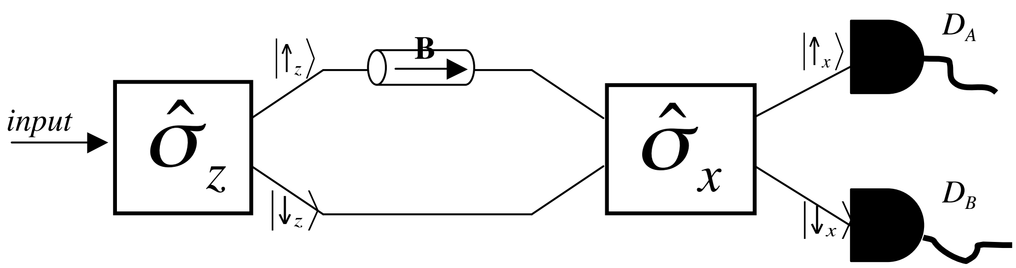
Physics 492: Quantum Mechanics II
Problem Set #6
Due: Thursday, April 1, 2004

Problem 1: Mixed states vs. pure states and interference (10 points)
A "spin-interferometer" is shown below



Spin-1/2 electrons are prepared in a given state (pure or mixed) are separated in two paths by a Stern-Gerlach apparatus (gradient field along z). In one path the particle passes through a solenoid, with a uniform magnetic field along the x-axis. The two paths are then recombined, sent through another Stern-Gerlach with gradient along x, and the particles are counted in detectors in the two emerging ports.

The strength of the magnetic field is chosen so that $\Omega t = \phi$, for some phase ϕ, where $\Omega = 2\mu_B B/\hbar$ is the Larmor frequency and t is the time spent inside the solenoid.

(a) Plot the probability of electrons arriving at detector B as a function of ϕ for the following pure state inputs: $(i)\, |\uparrow_z\rangle$, $(ii)\, |\uparrow_x\rangle$, $(iii)\, |\downarrow_x\rangle$.

(b) Repeat part (a) for the following mixed state inputs

$(i)\ \hat{\rho} = \frac{1}{2}|\uparrow_z\rangle\langle\uparrow_z| + \frac{1}{2}|\downarrow_z\rangle\langle\downarrow_z|$, $(ii)\ \hat{\rho} = \frac{1}{2}|\uparrow_x\rangle\langle\uparrow_x| + \frac{1}{2}|\downarrow_x\rangle\langle\downarrow_x|$, $(iii)\ \hat{\rho} = \frac{1}{3}|\uparrow_z\rangle\langle\uparrow_z| + \frac{2}{3}|\downarrow_z\rangle\langle\downarrow_z|$.

Comment on your results.

Problem 2: The spin singlet (10 points)
Consider the entangled state of two spins,

$$|\Psi_{AB}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle_A \otimes |\downarrow_z\rangle_B - |\downarrow_z\rangle_A \otimes |\uparrow_z\rangle_B\right).$$

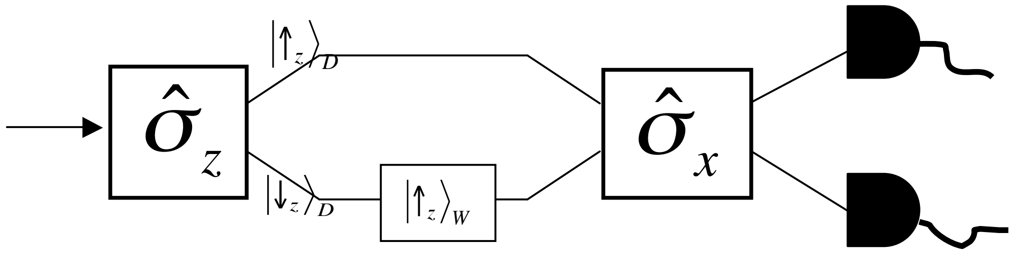
(a) Show that (up to a phase) $\frac{1}{\sqrt{2}}\left(|\uparrow_n\rangle_A \otimes |\downarrow_n\rangle_B - |\downarrow_n\rangle_A \otimes |\uparrow_n\rangle_B\right) = |\Psi_{AB}\rangle$, where $|\uparrow_n\rangle, |\downarrow_n\rangle$ are spin spin-up and spin-down states along the direction $\mathbf{e}_n$, discussed in P.S. #5. Interpret this result.

(b) Show that $\langle\Psi_{AB}|\hat{\sigma}_n \otimes \hat{\sigma}_{n'}|\Psi_{AB}\rangle = -\mathbf{e}_n \cdot \mathbf{e}_{n'}$

Problem 3: Which-path information, Entanglement, and Decoherence

We have discussed the rough rule of thumb encapsulated in Bohr's "Complementarity Principle": If we can determine which path a particle takes in an interferometer then we do not observe quantum interference fringes. But how does this arise

Consider the interferometer analogous to the one in Problem 1:



Into one arm of the interferometer we place a "which-way" detector in the form of another spin-1/2 particle prepared in the state $|\uparrow_z\rangle_W$. If the electron which travels through the interferometer, and ultimately detected (denoted *D)*, interacts with the "which-way" detector, the which-way electron flips the spin $|\uparrow_z\rangle_W \Rightarrow |\downarrow_z\rangle_W$.

(a) The electron *D* is initially prepared in the state $|\uparrow_x\rangle_D = \left(|\uparrow_z\rangle_D + |\downarrow_z\rangle_D\right)/\sqrt{2}$. Show that before detection, the two electrons *D* and *W* are in the entangled state

$$|\Psi_{DW}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle_D|\uparrow_z\rangle_W + |\downarrow_z\rangle_D|\downarrow_z\rangle_W\right).$$

(b) Only the electron *D* is detected. Show that its "marginal state", ignoring the electron *W,* is the completely mixed state,

$$\hat{\rho}_D = \frac{1}{2}|\uparrow_z\rangle_D\langle\uparrow_z| + \frac{1}{2}|\downarrow_z\rangle_D\langle\downarrow_z|$$

As you showed in Problem 1b, this state shows *no interference* between $|\uparrow_z\rangle$ and $|\downarrow_z\rangle$. Thus, the which-way detector *removes the coherence* between states that existed in the input.

(c) **Extra Credit:** Suppose now the which way detector does not function perfectly and acts the not completely flip the spin, but rotate it by an angle θ about so that,

$$|\uparrow_z\rangle_W \Rightarrow |\uparrow_\theta\rangle_W = \cos(\theta/2)|\uparrow_z\rangle_W + \sin(\theta/2)|\downarrow_z\rangle_W.$$

Show that in this case the marginal state is

$$\hat{\rho}_D = \frac{1}{2}\left(|\uparrow_z\rangle_D\langle\uparrow_z| + |\downarrow_z\rangle_D\langle\downarrow_z| + \cos(\theta/2)|\uparrow_z\rangle_D\langle\downarrow_z| + \cos(\theta/2)|\downarrow_z\rangle_D\langle\uparrow_z|\right).$$

Comment on the limits $\theta \to 0$ and $\theta \to \pi$.