Physics 492

Problem Set #6

Solutions

Problem 1: Postponed

Problem 2: The Spin Singlet

$$|\Psi\rangle_{singlet} = \frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle_A \otimes |\downarrow_z\rangle_B - |\downarrow_z\rangle_A \otimes |\uparrow_z\rangle_B\right)$$

This state has zero total angular momentum. It is thus rotationally invariant, and we expect it to look the same, irrespective of the quantization axis.

Recall from problem set #5

$$|\uparrow_n\rangle = \cos\left(\frac{\theta}{2}\right)|\uparrow_z\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|\downarrow_z\rangle$$

$$|\downarrow_n\rangle = \sin\left(\frac{\theta}{2}\right)|\uparrow_z\rangle - e^{i\phi}\cos\left(\frac{\theta}{2}\right)|\downarrow_z\rangle$$

are spin-up and spin-down along an arbitrary quantization axis defined by polar angles θ, ϕ

Consider then

$$\frac{1}{\sqrt{2}}\left(|\uparrow_n\rangle|\downarrow_n\rangle - |\downarrow_n\rangle|\uparrow_n\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\Big(\left(\cos\frac{\theta}{2}|\uparrow_z\rangle + e^{i\phi}\sin\frac{\theta}{2}|\downarrow_z\rangle\right)\otimes\left(\sin\frac{\theta}{2}|\uparrow_z\rangle - e^{i\phi}\cos\frac{\theta}{2}|\downarrow_z\rangle\right)$$
$$-\left(\sin\frac{\theta}{2}|\uparrow_z\rangle - e^{i\phi}\cos\frac{\theta}{2}|\downarrow_z\rangle\right)\otimes\left(\cos\frac{\theta}{2}|\uparrow_z\rangle + e^{i\phi}\sin\frac{\theta}{2}|\downarrow_z\rangle\right)\Big)$$

$$= \frac{-e^{i\phi}}{\sqrt{2}}\left(\left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right)|\uparrow_z\rangle|\downarrow_z\rangle - \left(\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}\right)|\downarrow_z\rangle|\uparrow_z\rangle\right)$$

$\therefore \quad \frac{1}{\sqrt{2}}\left(|\uparrow_n\rangle|\downarrow_n\rangle - |\downarrow_n\rangle|\uparrow_n\rangle\right)$

$= -e^{i\phi}\,|\Psi_{singlet}\rangle$

$\Rightarrow$ Up to an overall phase (which is irrelevant), the singlet is the same for all quantization axes.

(b) Consider the correlation function

$$\langle\Psi|\,\hat{\sigma}_n \otimes \hat{\sigma}_{n'}\,|\Psi\rangle_{singlet}$$

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle\otimes|\downarrow_z\rangle - |\downarrow_z\rangle\otimes|\uparrow_z\rangle\right)$$

$$\Rightarrow \hat{\sigma}_n\otimes\hat{\sigma}_{n'}\,|\Psi\rangle = \frac{1}{\sqrt{2}}\Big(\hat{\sigma}_n|\uparrow_z\rangle\otimes\hat{\sigma}_{n'}|\downarrow_z\rangle - \hat{\sigma}_n|\downarrow_z\rangle\otimes\hat{\sigma}_{n'}|\uparrow_z\rangle\Big)$$

$$\Rightarrow \langle\Psi|\,\hat{\sigma}_n\otimes\hat{\sigma}_{n'}\,|\Psi\rangle$$

$$= \frac{1}{2}\Big(\langle\uparrow_z|\hat{\sigma}_n|\uparrow_z\rangle\langle\downarrow_z|\hat{\sigma}_{n'}|\downarrow_z\rangle - \langle\downarrow_z|\hat{\sigma}_n|\uparrow_z\rangle\langle\uparrow_z|\hat{\sigma}_{n'}|\downarrow_z\rangle - \langle\uparrow_z|\hat{\sigma}_n|\downarrow_z\rangle\langle\downarrow_z|\hat{\sigma}_{n'}|\uparrow_z\rangle + \langle\downarrow_z|\hat{\sigma}_n|\downarrow_z\rangle\langle\uparrow_z|\hat{\sigma}_{n'}|\uparrow_z\rangle\Big)$$

Now, ~~...~~ from P.S. #5

$$\hat{\sigma}_n \doteq \begin{array}{c} \begin{bmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{bmatrix} \end{array} \begin{array}{l} \langle\uparrow_z| \\ \langle\downarrow_z| \end{array}$$

$$\Rightarrow \langle\Psi|\hat{\sigma}_n \otimes \hat{\sigma}_{n'}|\Psi\rangle$$

$$= \frac{1}{2}\Big(-\cos\theta\cos\theta' - e^{i(\phi-\phi')}\sin\theta\sin\theta'$$

$$- e^{-i(\phi-\phi')}\sin\theta\sin\theta' - \cos\theta\cos\theta'\Big)$$

$$= -\left(\cos\theta\cos\theta' + \cos(\phi-\phi')\sin\theta\sin\theta'\right)$$

Now $\vec{e}_n = \cos\theta\,\vec{e}_z + \sin\theta\,(\cos\phi\,\vec{e}_x + \sin\phi\,\vec{e}_y)$

$\vec{e}_{n'} = \cos\theta'\,\vec{e}_z + \sin\theta'\,(\cos\phi'\,\vec{e}_x + \sin\phi'\,\vec{e}_y)$

$$\Rightarrow \vec{e}_n \cdot \vec{e}_{n'} = \cos\theta\cos\theta' + \sin\theta\sin\theta'\,(\cos\phi\cos\phi' + \sin\phi\sin\phi')$$

$$= \cos\theta\cos\theta' + \sin\theta\sin\theta'\,(\cos(\phi-\phi'))$$

$$\Rightarrow \boxed{\langle\Psi|\hat{\sigma}_n \otimes \hat{\sigma}_{n'}|\Psi\rangle = -\vec{e}_n \cdot \vec{e}_{n'}}$$

These are the famous correlations of Bell's inequalities which <u>cannot</u> <u>be</u> captured in a local hidden variable theory.

Problem 3:

(a) The "which-way" ~~detec~~ detector works such that:

If $|\psi\rangle_D = |\uparrow_z\rangle$ nothing happens to $|\phi\rangle_W$

$|\psi\rangle_D = |\downarrow_z\rangle$ $\quad |\phi\rangle_W = |\uparrow_z\rangle_W \longrightarrow |\downarrow_z\rangle_W$

As a composite system

$$|\uparrow_z\rangle_D |\uparrow_z\rangle_W \Longrightarrow |\uparrow_z\rangle_D |\uparrow_z\rangle_W$$

$$|\downarrow_z\rangle_D |\uparrow_z\rangle_W \Longrightarrow |\downarrow_z\rangle_D |\downarrow_z\rangle_W$$

Consider then a state of electron-D which is in a ~~sa~~ superposition of up + down. The composite system before D and W interact

$$|\uparrow_x\rangle_D |\uparrow_z\rangle_W = \frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle_D + |\downarrow_z\rangle_D\right)|\uparrow_z\rangle_W$$

$$= \frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle_D |\uparrow_z\rangle_W + |\downarrow_z\rangle_D |\uparrow_z\rangle_W\right)$$

$$\underset{\text{After interaction}}{\Longrightarrow} \frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle_D |\uparrow_z\rangle_W + |\downarrow_z\rangle_D |\downarrow_z\rangle_W\right)$$

an entangled state!

(b) Given the state of the composite

$$|\Psi_{DW}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle_D |\uparrow_z\rangle_W + |\downarrow_z\rangle_D |\downarrow_z\rangle_W \right)$$

what is the state of D alone?

To answer this, consider the marginal probability distribution for some observable

$$P_{m_D} = \sum_{m_W} |\langle m_D, m_W | \Psi_{DW}\rangle|^2$$

$$= \frac{1}{2}\sum_{m_W} \left| \langle m_D|\uparrow_z\rangle_D \langle m_W|\uparrow_z\rangle_W + \langle m_D|\downarrow_z\rangle_D \langle m_W|\downarrow_z\rangle_W \right|^2$$

$$= \frac{1}{2} |\langle m_D|\uparrow_z\rangle_D|^2 \sum_{m_W} |\langle m_W|\uparrow_z\rangle_W|^2$$

$$+ \frac{1}{2} |\langle m_D|\downarrow_z\rangle_D|^2 \sum_{m_W} |\langle m_W|\downarrow_z\rangle_W|^2$$

$$+ \frac{1}{2} {}_D\langle\downarrow_z|m_D\rangle\langle m_D|\uparrow_z\rangle_D \sum_{m_W} {}_W\langle\downarrow_z|m_W\rangle\langle m_W|\uparrow_z\rangle_W$$

$$+ \frac{1}{2} \langle\uparrow_z|m_D\rangle\langle m_D|\downarrow_z\rangle_D \sum_{m_W} {}_W\langle\uparrow_z|m_W\rangle\langle m_W|\downarrow_z\rangle_W$$

<u>Aside:</u> $\sum_{m_W} |\langle m_W|\uparrow_z\rangle_W|^2 = \sum_{m_W} |\langle m_W|\downarrow_z\rangle_W|^2 = \underline{1}$

$$\sum_{m_W} {}_W\langle\uparrow_z|m_W\rangle\langle m_W|\downarrow_z\rangle_W = \sum_{m_W} {}_W\langle\downarrow_z|m_W\rangle\langle m_W|\uparrow_z\rangle_W$$

$$= 0$$

$$\therefore \; P_{m_D} = \frac{1}{2}\left|\langle m_D | \uparrow_z \rangle\right|^2 + \frac{1}{2}\left|\langle m_D | \downarrow_z \rangle\right|^2$$

This probability distribution is a <u>statistical</u> <u>mixture</u> of probabilities

arising from two different state $|\uparrow_z\rangle$ and $|\downarrow_z\rangle$

$\Rightarrow$ The marginal state is a statistical mixture of pure states

$$\boxed{\hat{\rho}_D = \frac{1}{2} |\uparrow_z\rangle_D \langle \uparrow_z| + \frac{1}{2} |\downarrow_z\rangle_D \langle \downarrow_z|}$$

(c) We now consider a less perfect "Which-way" detector which functions as

$$|\uparrow_z\rangle_D \, |\uparrow_z\rangle_W \Longrightarrow |\uparrow_z\rangle_D \, |\uparrow_z\rangle_W$$

$$|\downarrow_z\rangle_D \, |\uparrow_z\rangle_W \Longrightarrow | \quad \rangle_D \, |\uparrow_\theta\rangle_W$$

where $|\uparrow_\theta\rangle = \cos\frac{\theta}{2} |\uparrow_z\rangle + \sin\frac{\theta}{2} |\downarrow_z\rangle$

Note $\theta = 0 \Rightarrow$ No interaction

$\theta = \pi \Rightarrow$ Full spin flip

Again we calculated the joint state

$$|\uparrow_x\rangle_D |\uparrow_z\rangle_W \Longrightarrow \frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle_D |\uparrow_z\rangle_W + |\downarrow_z\rangle_D |\uparrow_\theta\rangle_W \right)$$

The marginal prob. distribution for D

$$P_{m_D} = \frac{1}{2}\sum_{m_W} \left| \langle m_D|\uparrow_z\rangle \langle m_W|\uparrow_W\rangle + \langle m_D|\downarrow_z\rangle_D \langle m_W|\uparrow_\theta\rangle_W \right|^2$$

$$= \frac{1}{2}|\langle m_D|\uparrow_z\rangle|^2 \sum_{m_W} |\langle m_W|\uparrow_z\rangle|^2$$

$$+ \frac{1}{2}|\langle m_D|\downarrow_z\rangle|^2 \sum_{m_W} |\langle m_W|\uparrow_\theta\rangle_W|^2$$

$$+ \frac{1}{2}\langle m_D|\uparrow_z\rangle_D \langle \downarrow_z|m_D\rangle \sum_{m_W} \langle \uparrow_\theta|m_W\rangle \langle m_W|\uparrow_z\rangle$$

$$+ \frac{1}{2}\langle m_D|\downarrow_z\rangle_D \langle \downarrow_z|m_D\rangle \sum_{m_W} \langle \uparrow_z|m_W\rangle \langle m_W|\uparrow_\theta\rangle$$

Now $\sum_{m_W} |\langle m_W|\uparrow_z\rangle_W|^2 = \sum_m |\langle m_W|\uparrow_\theta\rangle_W|^2 = 1$

$$\sum_{m_W} \langle \uparrow_\theta|m_W\rangle \langle m_W|\uparrow_z\rangle = \langle \uparrow_\theta|\uparrow_z\rangle = \cos\frac{\theta}{2}$$

$$\sum_{m_W} \langle \uparrow_z|m_W\rangle \langle m_W|\uparrow_\theta\rangle = \langle \uparrow_z|\uparrow_\theta\rangle = \cos\frac{\theta}{2}$$

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$$\Rightarrow P_{m_D} = \frac{1}{2}\left|\langle m_D|\uparrow_z\rangle_D\right|^2 + \frac{1}{2}\left|\langle m_D|\downarrow_z\rangle_D\right|^2$$

$$+ \frac{\cos\theta}{2}\ \langle m_D|\uparrow_z\rangle_D\langle\downarrow_z|m_D\rangle$$

$$+ \frac{\cos\theta}{2}\ \langle m_D|\downarrow_z\rangle_D\langle\uparrow_z|m_D\rangle$$

This probability distribution arises from a density operator

$$P_{m_D} = \langle m_D|\hat{\rho}_D|m_D\rangle$$

where
$$\boxed{\hat{\rho}_D = \frac{1}{2}|\uparrow_z\rangle_D\langle\uparrow_z| + \frac{1}{2}|\downarrow_z\rangle_D\langle\downarrow_z| + \frac{\cos\theta}{2}\left(|\uparrow_z\rangle_D\langle\downarrow_z| + |\downarrow_z\rangle_D\langle\uparrow_z|\right)}$$

in matrix form the density matrix is

$$\hat{\rho}_D \doteq \frac{1}{2}\begin{array}{c} \quad |\uparrow_z\rangle \quad\quad |\downarrow_z\rangle \\ \begin{bmatrix} 1 & \cos\theta/2 \\ \cos\frac{\theta}{2} & 1 \end{bmatrix}\end{array}$$

<u>Note</u>: $\theta \to \pi$, off diagonal elements vanishes
$\Rightarrow$ completely mixed state

$\theta \to 0$, off diagonal elements maximum
$\Rightarrow$ pure state

Upshot:

Feynman stated the rule:

"If two processes are in principle distinguishable (i.e. we can perform a measurement on some subsystem to see which process took place) then we add probabilities for the processes.

If two process are indistinguishable (i.e. there exist no information to measure) then we add probability amplitudes"

We see how this works with our "which way" detector. The paths are made distinguishable because $|\uparrow_z\rangle_W$ and $|\downarrow_z\rangle_W$ are orthogonal states. Thus we could, in principle measure electron-W to determined which path D had taken. The states $|\uparrow_z\rangle_W$ and $|\uparrow_\theta\rangle_W$ are not orthogonal for $0 \leq \theta < \pi$, and thus ~~[illegible]~~ interference is only partially removed until $\theta \to \pi$.