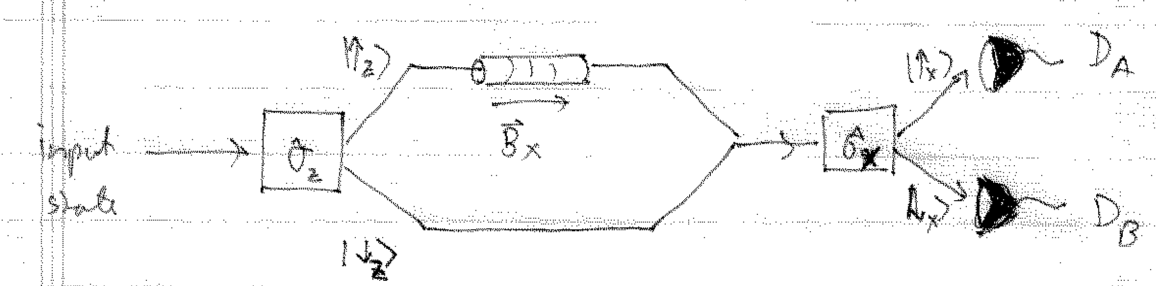
Physics 492 - Quantum II

P.S. #7 Solutions

Problem 1: Spin interferometer and coherence



(a) Consider an arbitrary pure state

$$|\psi\rangle = C_\uparrow |\uparrow_z\rangle + C_\downarrow |\downarrow_z\rangle$$

The spin $|\uparrow_z\rangle$ branch gets rotated by the operator $\hat{R}_x(\phi) = e^{-i\phi\frac{\hat{\sigma}_x}{2}} = \cos(\frac{\phi}{2})\,\mathbb{1} - i\sin(\frac{\phi}{2})\hat{\sigma}_x$

$\Rightarrow$ Right before the final beam splitter

$$|\psi'\rangle = C_\uparrow \hat{R}_x|\uparrow_z\rangle + C_\downarrow |\downarrow_z\rangle$$

$\Rightarrow$ Probability of detector D_B firing

$$P_{D_B} = |\langle\uparrow_x|\psi'\rangle|^2$$

$$= |C_\uparrow \langle\uparrow_x|\hat{R}_x|\uparrow_z\rangle + C_\downarrow \langle\uparrow_x|\downarrow_z\rangle|^2$$

(Next page)

Aside: • $\langle \uparrow_x | \hat{R}_x(\phi) | \uparrow_z \rangle =$

$$= \cos\left(\frac{\phi}{2}\right) \langle \uparrow_x | \uparrow_z \rangle - i \sin\left(\frac{\phi}{2}\right) \langle \uparrow_x | \hat{\sigma}_x | \uparrow_z \rangle$$

$$= \left(\cos\frac{\phi}{2} - i \sin\frac{\phi}{2}\right) \langle \uparrow_x | \uparrow_z \rangle \quad \left(\begin{array}{c}\text{using} \\ \langle \uparrow_x | \hat{\sigma}_x = \langle \uparrow_x | \end{array}\right)$$

$$= e^{-i\phi/2} \langle \uparrow_x | \uparrow_z \rangle$$

• Recall $|\uparrow_x\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle + |\downarrow_z\rangle\right)$

$$\Rightarrow \langle \uparrow_x | \uparrow_z \rangle = \langle \;\; | \downarrow_z \rangle = \frac{1}{\sqrt{2}}$$

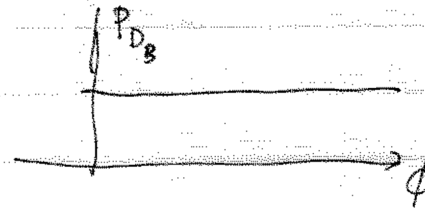
$$\therefore \quad P_{D_B} = \frac{1}{2} \left| c_\uparrow e^{-i\phi/2} + c_\downarrow \right|^2$$

$$\boxed{P_{D_B} = \frac{1}{2}\left(|c_\uparrow|^2 + |c_\downarrow|^2 + 2\,\mathrm{Re}\left(c_\uparrow c_\downarrow^* e^{-i\phi/2}\right)\right)}$$

Examples:

(i) $|\psi\rangle = |\uparrow_z\rangle \qquad c_\uparrow = 1, \; c_\downarrow = 0$

$$\Rightarrow \boxed{P_{D_B} = \frac{1}{2}}$$



(ii) $|\psi\rangle = |\downarrow_z\rangle \qquad c_\uparrow = 0, \; c_\downarrow = 1$

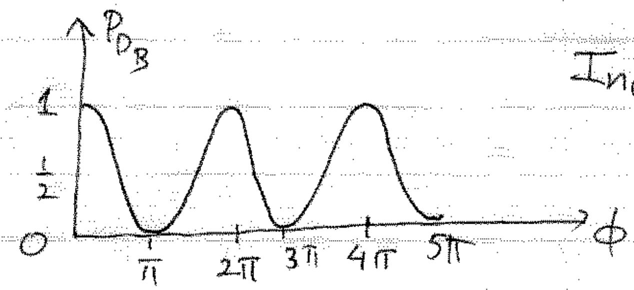
$$\Rightarrow \boxed{P_{D_B} = \frac{1}{2}}$$

These case show no interference because we know <u>which</u> path

(iii) $|\uparrow_x\rangle = \frac{1}{\sqrt{2}}|\uparrow_z\rangle + \frac{1}{\sqrt{2}}|\downarrow_z\rangle, \quad c_\uparrow = c_\downarrow = \frac{1}{\sqrt{2}}$

$$\Rightarrow P_{D_B} = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2} + \mathrm{Re}\left(e^{-i\phi/2}\right)\right)$$

$$= \frac{1+\cos\frac{\phi}{2}}{2} = \cos^2\phi$$

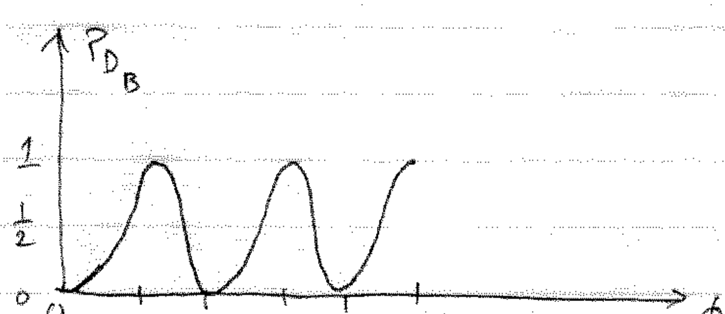


Interference "fringes"

(iv) $|\downarrow_x\rangle = \frac{1}{\sqrt{2}}|\uparrow_z\rangle - \frac{1}{\sqrt{2}}|\downarrow_z\rangle, \quad c_\uparrow = -c_\downarrow = \frac{1}{\sqrt{2}}$

$$\Rightarrow P_{D_B} = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2} - \mathrm{Re}\left(e^{-i\phi/2}\right)\right)$$

$$= \frac{1-\cos\frac{\phi}{2}}{2} = \sin^2\phi$$



(b) We consider now mixed states

Generally $\hat{\rho} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$

This is a <u>statistical mixture</u> ~~mixture~~ of states $\{|\psi_i\rangle\}$, not a coherent superposition of states. We should think of it "classically" $\rightarrow$ We have one of the set $\{|\psi_i\rangle\}$ we just don't know which one

$$\Rightarrow P_{D_B} = \sum_i P_i \, |\langle\uparrow_x|\psi_i\rangle|^2$$

Prob of $|\psi_i\rangle$

probability of $|\uparrow_x\rangle$ given $|\psi_i\rangle$

(i) $\hat{\rho} = \frac{1}{2}|\uparrow_z\rangle\langle\uparrow_z| + \frac{1}{2}|\downarrow_z\rangle\langle\downarrow_z|$

$$\text{Given } |\uparrow_z\rangle \Rightarrow P_{D_B}^{(\uparrow_z)} = \frac{1}{2}$$

$$|\downarrow_z\rangle \Rightarrow P_{D_B}^{(\downarrow_z)} = \frac{1}{2}$$

$$\Rightarrow P_{D_B}^{\text{total}} = P_{\uparrow_z} P_{D_B}(\uparrow_z) + P_{\downarrow_z} P_{D_B}(\downarrow_z)$$

$$= \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{2}$$

(ii) $\hat{\rho} = \frac{1}{2}|\uparrow_x\rangle\langle\uparrow_x| + \frac{1}{2}|\downarrow_x\rangle\langle\downarrow_x|$

$$\text{Given } |\uparrow_x\rangle \Rightarrow P_{D_B}^{(\uparrow_x)} = \cos^2\frac{\phi}{2}$$

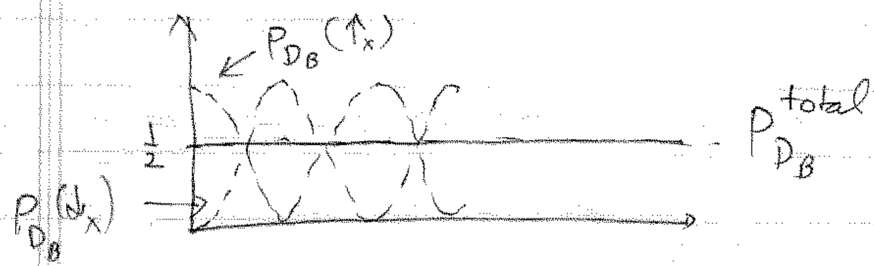
$$|\downarrow_x\rangle \Rightarrow P_{D_B}^{(\downarrow_x)} = \sin^2\phi/2$$

$$\Rightarrow P_{D_B}^{total} = P_{\uparrow_x} P_{D_B}(\uparrow_x) + P_{\downarrow_x} P_{D_B}(\downarrow_x)$$

$$= \frac{1}{2}\cos^2 \phi/2 + \frac{1}{2}\sin^2 \phi/2$$

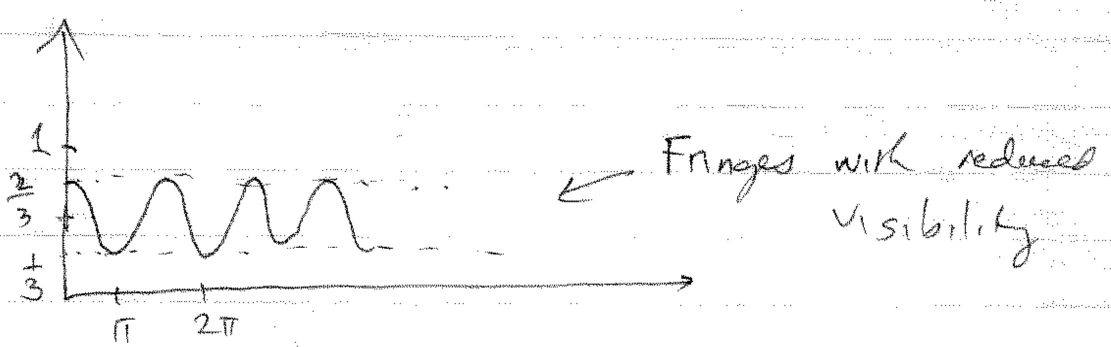
$$= \frac{1}{2}\left(\cos^2 \phi/2 + \sin^2 \phi/2\right) = \frac{1}{2}$$

$\Rightarrow$ For a completely mixed state the interference is removed



(iii) $\hat{\rho} = \frac{1}{3}|\uparrow_x\rangle\langle\uparrow_x| + \frac{2}{3}|\downarrow_x\rangle\langle\downarrow_x|$

$$P_{D_B} = \frac{1}{3}\cos^2\phi + \frac{2}{3}\sin^2\phi = \frac{1}{3} + \frac{1}{3}\sin^2\phi$$



A partial mixed state of two states which show interference shows fringes with reduced visibility.

Problem 2 Addition of spin and orbital angular momentum

An electron has both spin angular momentum, described by operator $\vec{S} = \hbar\vec{s}$ and orbital angular momentum described by operator $\vec{L} = \hbar\vec{l}$.

- We can describe the state in the "uncoupled representation" in terms of simultaneous eigenvectors of $\{\hat{l}^2, \hat{l}_z; \hat{s}^2, \hat{s}_z\}$

$|l m_l\rangle \otimes |s, m_s\rangle$ where $s = 1/2$ for electrons $\Rightarrow m_s = +1/2, -1/2$

Here we consider states with $l = 1 \Rightarrow m_l = 1, 0, -1$

- Alternately, we can use the "coupled representation" in terms of simultaneous eigenvectors of $\{\hat{j}^2, \hat{j}_z, \hat{l}^2, \hat{s}^2\}$

Where $\hat{\vec{j}} = \hat{\vec{l}} + \hat{\vec{s}} \Rightarrow \hat{j}_z = \hat{l}_z + \hat{s}_z$,

Eigenvectors: $|j m_j; l s\rangle$

Since $l = 1$, $s = 1/2$ is a common eigenvalue in both representations I will denote the short-hand for the eigenvectors

Uncoupled: $|m_l, m_s\rangle$ } l, s understood
Coupled: $|j, m_j\rangle$ } and common to both

Let us write $\hat{j}^2$ and $\hat{j}_z$ as matrices in the uncoupled basis

We need the relationship: $\hat{j}^2 = \hat{\vec{j}}\cdot\hat{\vec{j}} = \hat{\vec{l}}^2 + \hat{\vec{s}}^2 + 2\hat{l}_z\hat{s}_z + (\hat{l}_+\hat{s}_- + \hat{l}_-\hat{s}_+)$

$$\Rightarrow \hat{j}_z|m_l, m_s\rangle = (\hat{l}_z + \hat{s}_z)|m_l, m_s\rangle = (m_l + m_s)\,|m_l, m_s\rangle$$

and
$$\hat{j}^2|m_l, m_s\rangle = \{l(l+1) + s(s+1) + 2m_l m_s\}\,|m_l, m_s\rangle$$
$$+ \sqrt{l(l+1) - m_l(m_l+1)}\,\sqrt{s(s+1) - m_s(m_s-1)}\,|m_l+1, m_s-1\rangle$$
$$+ \sqrt{l(l+1) - m_l(m_l-1)}\,\sqrt{s(s+1) + m_s(m_s+1)}\,|m_l-1, m_s+1\rangle$$

Note: The uncoupled basis vectors $|m_\ell, m_s\rangle$ are already eigenvectors of $\hat{J}_z$, with eigenvalue $m_j = m_\ell + m_s$

Since $m_\ell = 1, 0, -1$ and $m_s = \frac{1}{2}, -\frac{1}{2}$, the possible values of m_j are $m_j = \pm 3/2$ and $\pm 1/2$, with a double degeneracy for $m_j = \pm 1/2$.

$$\begin{cases} m_j = 3/2 & \Rightarrow \; |m_\ell = 1, \; m_s = 1/2\rangle \\ m_j = 1/2 & \Rightarrow \; |m_\ell = 0, m_s = 1/2\rangle \; \text{ or } \; |m_\ell = 1, \; m_s = -1/2\rangle \\ m_j = -1/2 & \Rightarrow \; |m_\ell = 0, m_s = -1/2\rangle \; \text{ or } \; |m_\ell = -1, \; m_s = 1/2\rangle \\ m_j = -3/2 & \Rightarrow \; |m_\ell = -1, \; m_s = -1/2\rangle \end{cases}$$

We can simplify the calculation by ordering the basis into orthogonal-subspaces. Since the diagonalization of $\hat{J}^2$ cannot mix states with different m_j, we choose the ordered basis

$$|m_\ell, m_s\rangle = \{\underbrace{|1, \tfrac{1}{2}\rangle}_{m_j = 3/2}; \; \underbrace{|0, \tfrac{1}{2}\rangle, |1, -\tfrac{1}{2}\rangle}_{m_j = \frac{1}{2}}; \; \underbrace{|0, -\tfrac{1}{2}\rangle, |-1, \tfrac{1}{2}\rangle}_{m_j = -1/2}; \; \underbrace{|-1, -\tfrac{1}{2}\rangle}_{m_j = -3/2}\}$$

$$\Rightarrow \quad \hat{J}_z = \begin{bmatrix} \frac{3}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3}{2} \end{bmatrix}, \quad \hat{J}^2 = \begin{bmatrix} \frac{15}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{11}{4} & \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & \frac{7}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{11}{4} & \sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{2} & \frac{7}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{15}{4} \end{bmatrix}$$

Thus in each of the 2D subspaces with $m_j = 1/2$ and $m_j = -1/2$ we must diagonize the same matrix

$$M = \begin{bmatrix} \frac{11}{4} & \sqrt{2} \\ \sqrt{2} & \frac{7}{4} \end{bmatrix}$$

The secular equation for <u>both</u> $m_j = +1/2$ and $m_j = -1/2$ subspa[ce]

$$\det(\lambda \mathbb{1} - M) = (\lambda - \tfrac{11}{4})(\lambda - \tfrac{7}{4}) - 2 = 0$$

$$\Rightarrow \lambda^2 - \tfrac{9}{2}\lambda + \tfrac{45}{16} = 0$$

$$\Rightarrow \lambda = \tfrac{3}{4} \text{ or } \lambda = \tfrac{15}{4}$$

Remember, <u>eigenvalue</u> of $\hat{j}^2$ denoted $j(j+1)$ $\Rightarrow j = \frac{1}{2}$ or $\frac{3}{2}$

<u>Eigenvectors</u> $j = \frac{1}{2}$: $\begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \frac{a}{b} = -\frac{1}{\sqrt{2}}$

Normalized: $\begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\sqrt{\frac{2}{3}} \end{bmatrix}$ (Remember, this is a representation given an ordered basis)

$j = \frac{3}{2}$: $\begin{bmatrix} -1 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \frac{a}{b} = \sqrt{2}$: Normalized: $\begin{bmatrix} \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$

We thus have the following eigenvectors in the coupled representation:

$$j = 3/2 \begin{cases} |j = 3/2, m_j = +3/2\rangle = |m_\ell = 1, m_s = +1/2\rangle \\ |j = 3/2, m_j = +1/2\rangle = \sqrt{\frac{2}{3}}\,|m_\ell = 0, m_s = 1/2\rangle + \sqrt{\frac{1}{3}}\,|m_\ell = 1, m_s = -1/2\rangle \\ |j = 3/2, m_j = -1/2\rangle = \sqrt{\frac{2}{3}}\,|m_\ell = 0, m_s = -1/2\rangle + \sqrt{\frac{1}{3}}\,|m_\ell = -1, m_s = +1/2\rangle \\ |j = 3/2, m_j = -3/2\rangle = |m_\ell = -1, m_s = -1/2\rangle \end{cases}$$

$$j = 1/2 \begin{cases} |j = 1/2, m_j = +1/2\rangle = \sqrt{\frac{1}{3}}\,|m_\ell = 0, m_s = 1/2\rangle - \sqrt{\frac{2}{3}}\,|m_\ell = 1, m_s = -1/2\rangle \\ |j = 1/2, m_j = -1/2\rangle = \sqrt{\frac{1}{3}}\,|m_\ell = 0, m_s = -1/2\rangle - \sqrt{\frac{2}{3}}\,|m_\ell = -1, m_s = +1/2\rangle \end{cases}$$

Note:

- The possible eigenvalues of j range from $j_{max} = l+s = \frac{3}{2}$ to $j_{min} = |l-s| = \frac{1}{2}$ as expected
- The Clebsch-Gordan coefficients are now given

$$\langle j m_j | l m_l ; s m_s \rangle = \langle l m_l ; s m_s | j m_j \rangle$$

$$\Rightarrow \langle \tfrac{3}{2}\ \tfrac{3}{2} | 1 1 ; \tfrac{1}{2}\ \tfrac{1}{2} \rangle = \langle \tfrac{3}{2}, -\tfrac{3}{2} | 1 -1, \tfrac{1}{2} -\tfrac{1}{2} \rangle = 1$$

$$|\langle \tfrac{3}{2}\ \tfrac{1}{2} | 1 1, \tfrac{1}{2} -\tfrac{1}{2} \rangle| = |\langle \tfrac{3}{2}, -\tfrac{1}{2} | 1 -1, \tfrac{1}{2}\ \tfrac{1}{2} \rangle| = |\langle \tfrac{1}{2}\ \tfrac{1}{2} | 1 0, \tfrac{1}{2}\ \tfrac{1}{2} \rangle| = |\langle \tfrac{1}{2} -\tfrac{1}{2} | 1 0, \tfrac{1}{2} -\tfrac{1}{2} \rangle| = \sqrt{\tfrac{1}{3}}$$

$$|\langle \tfrac{3}{2}\ \tfrac{1}{2} | 1 0, \tfrac{1}{2}\ \tfrac{1}{2} \rangle| = |\langle \tfrac{3}{2}, -\tfrac{1}{2} | 1 0, \tfrac{1}{2} -\tfrac{1}{2} \rangle| = |\langle \tfrac{1}{2}\ \tfrac{1}{2} | 1 1, \tfrac{1}{2} -\tfrac{1}{2} \rangle| = |\langle \tfrac{1}{2}\ \tfrac{1}{2} | 1 -1, \tfrac{1}{2}\ \tfrac{1}{2} \rangle| = \sqrt{\tfrac{2}{3}}$$

We cannot assure the sign of CG coefficient is consistent with our phase convention by this method.

(b) We can easily find the ~~eigenstates~~ matrix elements of $\hat{\vec{l}} \cdot \hat{\vec{S}}$ by noting

$$\hat{\vec{j}}^2 = (\hat{\vec{l}} + \hat{\vec{s}}) \cdot (\hat{\vec{l}} + \hat{\vec{s}}) = \hat{\vec{l}}^2 + \hat{\vec{s}}^2 + 2\hat{\vec{l}} \cdot \hat{\vec{s}}$$

$$\Rightarrow \hat{\vec{l}} \cdot \hat{\vec{S}} = \frac{1}{2}\left(\hat{\vec{j}}^2 - \hat{\vec{l}}^2 - \hat{\vec{s}}^2\right)$$

$$\Rightarrow \langle j' m_j' ; l, s | \hat{\vec{l}} \cdot \hat{\vec{S}} | j m_j ; l s \rangle = \frac{1}{2}\left(j(j+1) - l(l+1) - s(s+1)\right) \delta_{jj'} \delta_{m_j m_j'}$$

$$\Rightarrow \langle j = \tfrac{1}{2}\ m_j ; l, s | \hat{\vec{l}} \cdot \hat{\vec{S}} | j = \tfrac{1}{2}, m_j, l, s \rangle = -2$$

$$\langle j = \tfrac{3}{2}, m_j ; l s | \hat{\vec{l}} \cdot \hat{\vec{S}} | j = \tfrac{3}{2}, m_j, l s \rangle = 1$$

$$= 0 \quad \text{otherwise}$$