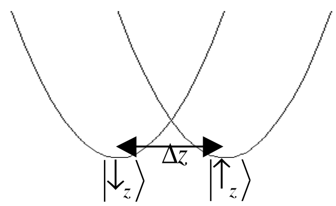
Physics 492: Quantum Mechanics II
Problem Set #10
Due: Thursday, April 29, 2004

Problem 1: Motion in spin dependent traps (10 Points)
Consider an electron moving in one dimension, in a spin-dependent trap as shown below:



If the electron is in spin-up (with respect to the *z*-axis), it is trapped in the right harmonic well, and if it is in spin-down (with respect to the *z*-axis), it is trapped in the left harmonic well. The Hamiltonian that governs its dynamics can be written as,

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega_{osc}^2 (\hat{z} - \Delta z/2)^2 \otimes |\uparrow_z\rangle\langle\uparrow_z| + \frac{1}{2} m\omega_{osc}^2 (\hat{z} + \Delta z/2)^2 \otimes |\downarrow_z\rangle\langle\downarrow_z|.$$

(a) What are the energy spectrum and stationary states of the system? What are the degeneracies of these states? Sketch an energy level diagram for the first three levels and label the degeneracies.

A small *constant* "transverse field" B_x is now added, with $|\mu_B B_x| << \hbar\omega_{osc}$.
(b) Qualitatively sketch how the energy plot in part (a) is modified.

(c) Now calculate the perturbed energy spectrum for this system.

(d) What are the new eigenstates in the ground state doublet? For Δz macroscopic, these are sometimes called Schrödinger cat states -- explain why.

Problem 2: The "Normal" Zeeman effect (10 points)
The Zeeman effect refers to the splitting of spectral lines due to the application of an applied magnetic field. It was observed first, before the invention of quantum mechanics, and explained in terms of classical electron response to electromagnetic fields.

(a) Consider an electron bound to a proton in a circular "planetary orbit" of radius *r* and frequency ω_0. This electron radiates electromagnetically at frequency ω_0. A weak magnetic field is applied, exerting a force much smaller than the binding. Given many atoms, on average, the field will be either normal to, or in the plane motion. Show that this leads to three spectral lines at frequencies, ω_0 and $\omega_0 \pm \Omega$, where $\Omega = \dfrac{eB}{2mc}$ is the classical Larmor frequency.

(b) Consider now the quantum model of hydrogen. Suppose our spectrometer cannot resolve the fine structure and we observe radiation on the $2p \to 1s$ transition (i.e. spin is irrelevant to our observation). Show that in the presence of a magnetic field, there are three spectral lines at the same frequencies: ω_0 and $\omega_0 \pm \Omega$. This is known a "normal Zeeman effect" since was explained by the classical physics at the time.

Problem 3: The "Anomalous" Zeeman effect (10 points)

Suppose now we take spin into consideration. Its effect will be observable in the Zeeman effect if our spectrometer can resolve the fine structure.

(a) Show that the Zeeman interaction Hamiltonian is

$\hat{H}_{Zeeman} = -\hat{\vec{\mu}} \cdot \mathbf{B} = -\mu_B \left(\hat{\mathbf{j}} + \hat{\mathbf{s}} \right) \cdot \mathbf{B}$, where $\mathbf{j} = \mathbf{l} + \mathbf{s}$ is the total electron angular momentum in units of $\hbar$. (Hint: Remember the *g*-factors for orbital and spin).

The total Hamiltonian is thus $\hat{H} = \hat{H}_{spinless} + \hat{H}_{FS} + \hat{H}_{Zeeman}$, where $\hat{H}_{spinless}$ is the hydrogen atom Hamiltonian in the absence of spin, $\hat{H}_{FS}$ Hamiltonian including spin-orbit coupling and relativistic corrections. When the Zeeman energy is much smaller than the fine-structure splitting, it is a perturbation. with $\hat{H}_{spinless} + \hat{H}_{FS}$ the "zeroth order" Hamiltonian.

(b) Consider then the Zeeman effect on the $2p_{1/2} \to 1s_{1/2}$ transition. Show that there are only *two* spectral lines (this is known as the anomalous Zeeman effect).

The Landé projection theorem (not proved here) says that when a vector operator (like $\hat{\vec{\mu}}$) is restricted is restricted to a subspace with total angular momentum **j**, is acts like the operator,

$$\hat{\vec{\mu}} = \left\langle \frac{\hat{\vec{\mu}} \cdot \hat{\mathbf{j}}}{\hat{\mathbf{j}}^2} \right\rangle \hat{\mathbf{j}} \text{ (on manifold with fixed j).}$$

(c) Use this to show that in first order perturbation theory, the applied magnetic leads to a splitting of the $2j+1$ degenerate sublevels of a given fine structure level to have energy,

$$E_{njlm_j} = E^{(0)}_{njl} + m_j \, (g_{jl} \mu_B B),$$

where $g_{jl} = 1 + \dfrac{j(j+1) - l(l+1) + 3/4}{2j(j+1)}$ is the "Landé *g*-factor".