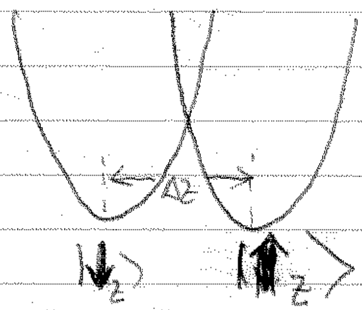
Physics 492: Quantum II

Problem Set #10 Solutions

Problem 1: Motion in spin dependent traps



Trap correlated with "internal state"

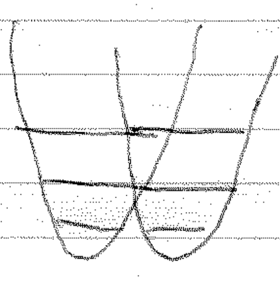
(a) In the absence of coupling between $|\Uparrow_z\rangle$ and $|\Downarrow_z\rangle$ there is a doubly degenerate spectrum

Energy levels $E_n = (n+\frac{1}{2})\hbar\omega \quad n = 0, 1, 2, 3, \ldots$

Two spin states $|\Uparrow_z\rangle$ and $|\Downarrow_z\rangle$

$\Rightarrow$ Two orthogonal quantum states for each energy level: $\{ |n\rangle_R \otimes |\Uparrow_z\rangle$ and $|n\rangle_L \otimes |\Downarrow_z\rangle \}$

where $|n_R\rangle$ and $|n_L\rangle$ are the Hermite-Gaussian wave packets centered at each well



energy levels on trap.

(b) We now turn on a small transverse magnetic field. The two states in each degenerate manifold are now coupled. This perturbation will <u>break</u> the degeneracy.

(Note: Not to scale, the splitting is $<<$ $\hbar\omega$)

(c) ~~Calculation~~ Calculation using degenerate perturbation theory.

Each level E_n is doubly degenerate.

let $|1\rangle \equiv |n\rangle_L \otimes |\downarrow_z\rangle$

$|2\rangle \equiv |n\rangle_R \otimes |\uparrow_z\rangle$

The perturbation Hamiltonian $\hat{H}_1 = \underbrace{(\mu_B B_x)}_{= \hbar\Omega/2} \hat{\sigma}_x$

We must diagonalized $\hat{H}_1$ is the basis $\{|1\rangle, |2\rangle\}$

<u>Note:</u> $\hat{H}_1$ does not couple the motional degree of freedom (only the spin)

$$\Rightarrow \quad \hat{H}_1 = \underbrace{\hbar\frac{\Omega}{2}}_{\text{Spin coupling}} \hat{\sigma}_x \otimes \hat{1}_{\text{motion}}$$

Matrix representation:

$$\hat{H}_1 \doteq \begin{bmatrix} \langle 1|\hat{H}_1|1\rangle & \langle 1|\hat{H}_1|2\rangle \\ \langle 2|\hat{H}_1|1\rangle & \langle 2|\hat{H}_1|2\rangle \end{bmatrix}$$

$$\langle 1|\hat{H}_1|1\rangle = \left(\langle \downarrow_z| \otimes \langle n_L|\right) \hat{H}_1 \left(|\downarrow_z\rangle \otimes |n_L\rangle\right)$$

$$= \underbrace{\langle \downarrow_z|\vec{\sigma}_x|\downarrow_z\rangle}_{=0} \langle n_L|n_L\rangle$$

Similarly $\langle 2|\hat{H}_1|2\rangle = 0$ since $\langle \uparrow_z|\vec{\sigma}_x|\uparrow_z\rangle = 0$

However,

$$\langle 1|\hat{H}_1|2\rangle = \left(\langle \downarrow_z| \otimes \langle n_L|\right) \hat{H}_1 \left(|\uparrow_z\rangle \otimes |n_R\rangle\right)$$

$$= \mu_B B_x \underbrace{\langle \downarrow_z|\vec{\sigma}_x|\uparrow_z\rangle}_{=1} \langle n_L|n_R\rangle$$

$$= \mu_B B_x \langle n_L|n_R\rangle$$

$$\langle 2|\hat{H}_1|1\rangle = \mu_B B_x \langle n_R|n_L\rangle$$

since Hermite Gaussians are real

$$= \mu_B B_x \langle n_L|n_R\rangle$$

Thus, $\hat{H}_1 \doteq \mu_B B_x \langle n_L | n_R \rangle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$\boxed{\begin{aligned} &\text{Eigenvalues} \quad E_{\pm}^{(1)} = \pm \mu_B B_x \langle n_L | n_R \rangle \\ &\quad \text{where} \quad \langle n_L | n_R \rangle = \int_{-\infty}^{\infty} dx \, \phi_n\left(x - \tfrac{\Delta z}{2}\right) \phi_n\left(x + \tfrac{\Delta z}{2}\right) \\ &\qquad\qquad = \text{Overlap of wave packets} \end{aligned}}$$

Note: Unlike the spin in free space, here the splitting between the spin states depends on the spatial overlap of the two wave packets. This is known as a tunneling splitting, as the particle must tunnel to the neighboring well to flip its spin.

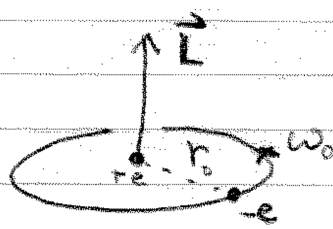
Note: the eigenvectors are entangled states

$$|\pm\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle \pm |2\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|n_L\rangle \otimes |\downarrow_z\rangle + |n_R\rangle \otimes |\uparrow_z\rangle\right)$$

Problem 2: The "Normal" Zeeman effect

(a) Classical electron view



electron orbits the proton at frequency ω_0.
It radiates E/M radiation @ frequency ω_0

For circular orbit, centripetal force is coulomb force

$$m\omega_0^2 r_0 = \frac{e^2}{r_0^2}$$

Now add an external magnetic field. In addition to the coulomb force, there is a small Lorentz force, $\vec{F} = e\frac{\vec{v}}{c} \times \vec{B}$

- For $\vec{B} \perp$ to plane $\quad F_r = \pm e\frac{v}{c}B_\perp = \pm e\frac{\omega}{c}r_0 B_\perp$

$\Rightarrow$ Centripetal force $\quad m\omega^2 r_0 = \frac{e^2}{r_0^2} \pm e\frac{\omega}{c}r_0 B_\perp$

$\uparrow$ $m\omega_0^2 r_0^2$

$\Rightarrow$ Let $\omega = \omega_0 + \Delta\omega$

$\uparrow$ shift $<< \omega_0$

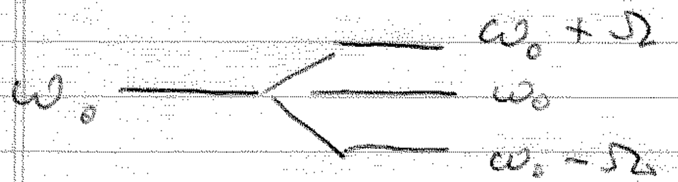
$$\Rightarrow \omega^2 \approx \omega_0^2 + 2\omega_0 \Delta\omega$$

$$\therefore 2m\omega_0 r_0 \Delta\omega_\pm \approx \pm e\frac{\omega_0 r_0}{c}B_\perp$$ (Next Page)

$$\Rightarrow \boxed{\Delta\omega_{\pm} \approx \pm \frac{e B_{\perp}}{2mc} = \pm \Omega \quad \text{For } \vec{B} \perp \text{ to } \vec{v}}$$

For $\vec{B}$ in the plane the orbits' angular frequency is not shifted $\Rightarrow \boxed{\Delta\omega_{\parallel} = 0}$

$\Rightarrow$ In the classical picture, the spectral line is split into three component



Where $\Omega = \dfrac{eB}{2mc}$

(b) Now consider $1s \leftrightarrow 2p$ transition in Hydrogen (excluding spin in the description)

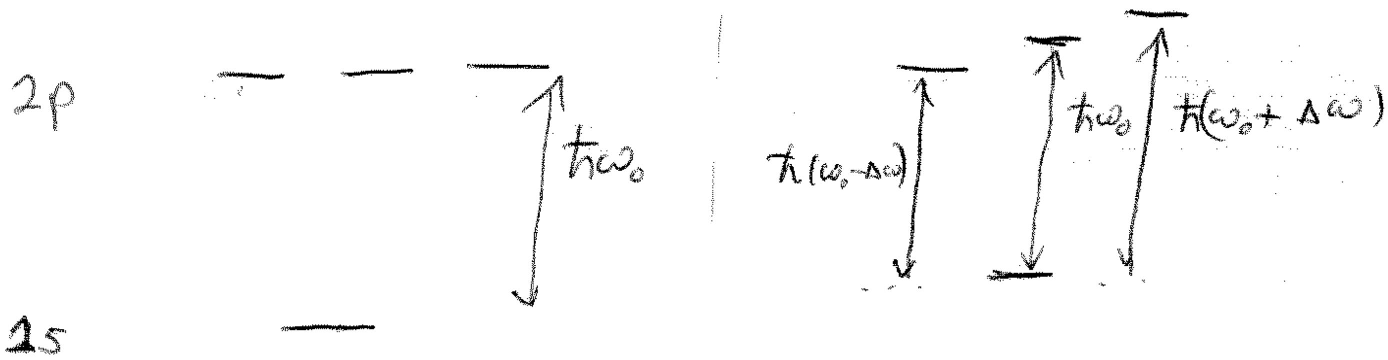
Zeeman interaction $\hat{H} = -\hat{\vec{\mu}} \cdot \vec{B} = \mu_B B \hat{l}_z$

(using $\hat{\vec{\mu}} = -\mu_B \hat{\vec{L}}/\hbar$ and $\vec{B} = B\,\vec{e}_z$)

$\Rightarrow$ "Zeeman shift" $\Delta E_{m_l} = \mu_B B m_l$

"magnetic quantum #"

$\Rightarrow$ Unperturbed levels | Perturbed by $\vec{B}$ field



three spectral lines
@ $\omega = \omega_0 \,,\, \omega_0 \pm \Delta\omega$

Shift $\Delta\omega = \pm \dfrac{\mu_B B}{\hbar} = \pm \dfrac{eB}{2mc}$ as in Classical model

Thus, in the absence of spin, the Zeeman effect can be explained <u>Classically</u>

Problem 3: The "Anamalous" Zeeman effect

Taking spin into ~~consid~~ consideration

(a) Interaction Hamiltonian $\hat{H}_{\text{Zeeman}} = -\hat{\vec{\mu}} \cdot \vec{B}$

$$\hat{\vec{\mu}} = -\mu_B \left(\underbrace{g_\ell \hat{\vec{\ell}} + g_s \hat{\vec{s}}}_{\text{g-factors}} \right)$$

$g_\ell = 1$ $\qquad g_s = 2 \quad \Rightarrow \hat{\vec{\mu}} = -\mu_B(\hat{\vec{\ell}} + 2\hat{\vec{s}})$

$$= -\mu_B(\hat{\vec{j}} + \hat{\vec{s}})$$

$$\Rightarrow \boxed{\begin{array}{c} \hat{H}_{\text{Zeeman}} = \mu_B(\hat{\vec{j}} + \hat{\vec{s}}) \cdot \vec{B} = \mu_B B(\hat{j}_z + \hat{s}_z) \\ \text{where } \vec{B} = B\vec{e}_z \end{array}}$$

(b) When the Zeeman energy $\mu_B B$ is much smaller than the fine-structure splitting, we ~~~~ can treat it as a perturbation to these states.

Consider $1s_{1/2} \leftrightarrow 2p_{1/2}$ transition

"Good" quantum numbers $|n\, j\, m_j\, l\, s\rangle$

$$1s_{1/2} = |n=1\; j=1/2\; m_j=\pm\tfrac{1}{2},\; l=0,\; s=1/2\rangle$$

(doubly degenerate)

$$2p_{1/2} = |n=2,\; j=1/2,\; m_j=\pm 1/2,\; l=1,\; s=1/2\rangle$$

(doubly degenerate)

<u>Unperturbed</u> W/ B-field

there are actually <u>four</u> spectral lines (contrary to the statement of problem - I forgot ground state splitting).

$$\omega = \omega_0 \pm \frac{(\Delta E_e + \Delta E_g)}{\hbar}$$

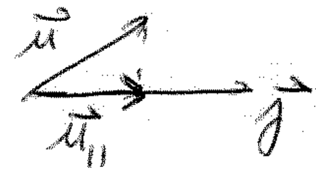
and

$$\omega = \omega_0 \pm \left(\frac{\Delta E_e - \Delta E_g}{\hbar}\right)$$

(c) We now use the Landé projection theorem.

$$\hat{\vec{\mu}} \Rightarrow \left\langle \frac{\hat{\vec{\mu}}\cdot\hat{\vec{J}}}{\hat{\vec{J}}^2} \right\rangle \hat{\vec{J}}$$

(Think of classical projection onto $\vec{J}$)



The fine-structure states are eigenstates of $\hat{H}_{Zeeman}$ with this theorem.

$$\hat{\vec{\mu}} = -\mu_B B\,(\hat{\vec{J}} + \hat{\vec{S}}) \Longrightarrow -\mu_B B \left\langle \frac{\hat{\vec{J}}^2 + \hat{\vec{S}}\cdot\hat{\vec{J}}}{\hat{\vec{J}}^2} \right\rangle \hat{\vec{J}}$$

Note $\hat{\vec{J}} = \hat{\vec{L}} + \hat{\vec{S}} \Rightarrow \hat{\vec{L}} = \hat{\vec{J}} - \hat{\vec{S}}$

$$\Rightarrow \hat{\vec{L}}^2 = \hat{\vec{J}}^2 + \hat{\vec{S}}^2 + 2\hat{\vec{J}}\cdot\hat{\vec{S}}$$

$$\Rightarrow \hat{\vec{S}}\cdot\hat{\vec{J}} = \frac{\hat{\vec{J}}^2 - \hat{\vec{L}}^2 + \hat{\vec{S}}^2}{2}$$

$$\Rightarrow \hat{\vec{\mu}} \Longrightarrow -\mu_B \left\langle 1 + \frac{\hat{\vec{J}}^2 - \hat{\vec{L}}^2 + \hat{\vec{S}}^2}{2} \right\rangle \vec{J}$$

Hamiltonian:

$$\hat{H}_{zeeman} = \mu_B B \left\langle 1 + \frac{\hat{J}^2 - \hat{\ell}^2 + \hat{S}^2}{2} \right\rangle \hat{J}_z$$

Eigenstates $|n, j m_j \ell s\rangle$ $\qquad s = 1/2$

$\Rightarrow$ Eigenvalues $E_{m_j} = \mu_B B \underbrace{\left(1 + \frac{j(j+1) - \ell(\ell+1) - \frac{3}{4}}{2} \right)}_{g_{j\ell}} m_j$

$g_{j\ell}$ ↖ Landé g-factor

Thus the energy levels ~~[illegible]~~

$$\boxed{E_{nj\ell m_j} = E^{(0)}_{nj\ell} + m_j (g_{j\ell} \mu_B B)}$$