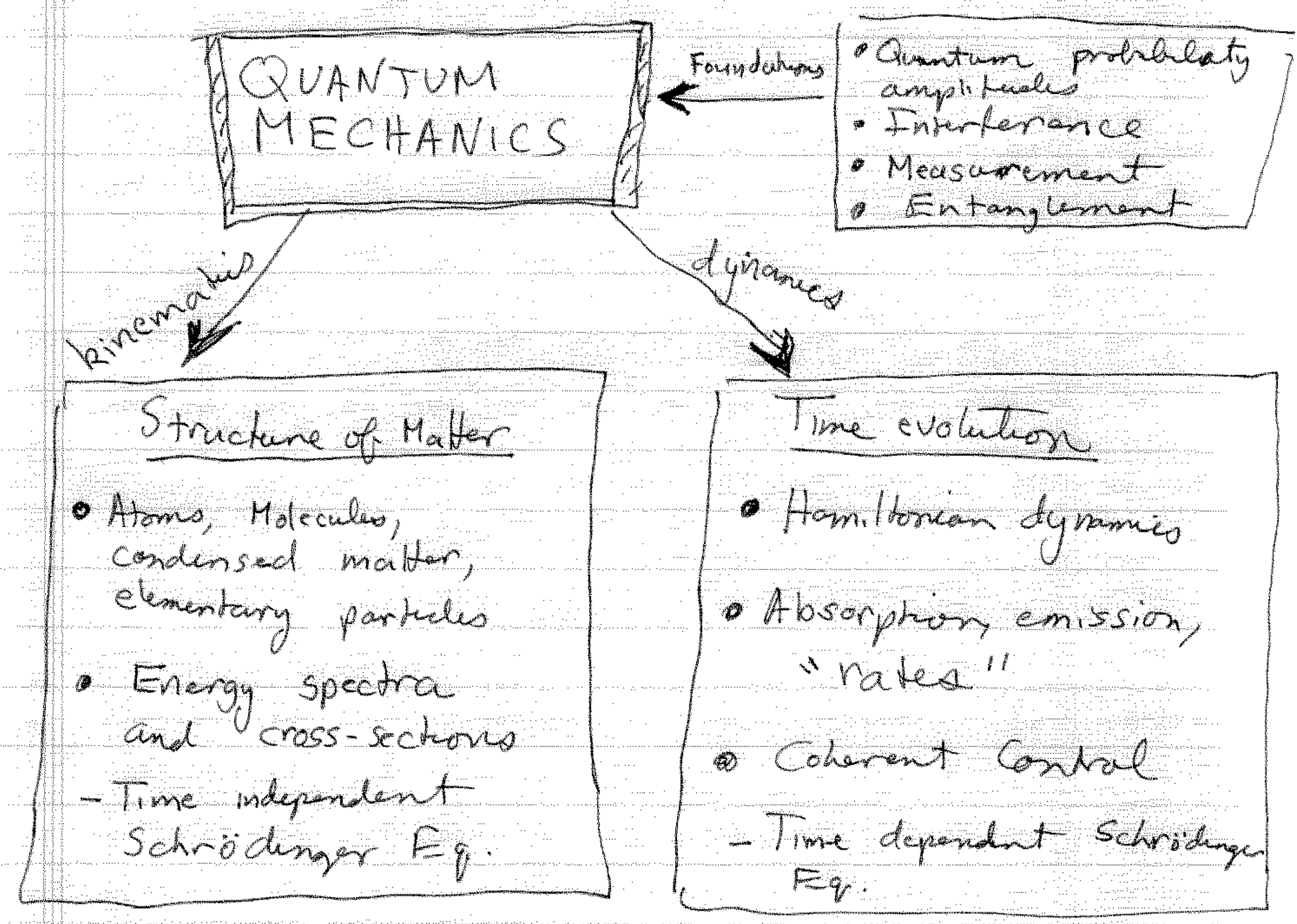
Physics 492: Quantum Mechanics II

Lecture 1: Review of Quantum theory

Quantum Mechanics, what's it about?



Traditional studies of quantum mechanics have focused primarily on kinematics, with minimal emphasis on dynamics (save for basic "rate processes"). The foundations are not usually given much thought after the basic axioms are established. Today interest in foundations has resurged with possible application to information processing. Observation of coherent dynamics now possible in laboratory.

Basic structure of quantum theory

Quantum theory is <u>fundamentally statistical</u>

$\Rightarrow$ Events cannot be predicted with certainty, but instead we can only associate probabilities to their outcomes.

<u>Baysean Probabilities</u>: Probability "subjective", assigned based on "state of knowledge"

Classical Probability

Random variable, a, Probability distribution $P(a)$

Classical reasoning: $P(a, b) = P(a|b)\, P(b)$

$$P(a, \vee\{b_i\}) = \sum_i P(a|b_i)\, P(b_i)$$

Probability a is true and one of $\{b_i\}$ is true

($\{b_i\}$ = Distinct possible alternatives)

Sum of probabilities for alternatives

$P(a|b_i)$ = Probability a is true given b_i is true

$P(b_i)$ = Probability b_i is true.

Basic logic

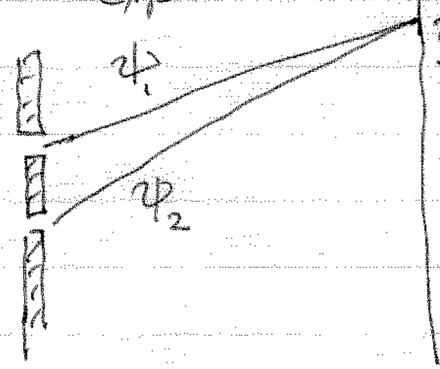
Quantum State: Wave function $\psi(x)$
$= \text{state of knowledge}$

Used to predict outcome of experiments

Born rule: $|\psi(x)|^2 =$ Probability (density) to find particle near x

Aside:
- $\psi(x)$ is a complex number
- $\psi(x)$ and $e^{i\phi}\psi(x)$ are same state
- Generally $\psi(x)$ and $\alpha\psi(x)$ same state
- Normalization $\int dx\, |\psi(x)|^2 = 1$

Double slit experiment: Superposition Principle



$\Psi = \psi_1 + \psi_2$

two possible paths lead to event

$$|\psi|^2 = |\psi_1 + \psi_2|^2 = (\psi_1 + \psi_2)^* (\psi_1 + \psi_2)$$
$$= |\psi_1|^2 + |\psi_2|^2 + (\psi_1^* \psi_2 + \psi_2^* \psi_1)$$

$\Rightarrow P = P_1 + P_2 +$ Interference terms

Indistinguishable paths interfere

More general description of state

$|\psi\rangle$ = vector in Hilbert space $\mathcal{H}$

$\mathcal{H}$ = abstract vector space with innerproduct (like dot product for vectors in 3D)

$$\langle \psi_1 | \psi_2 \rangle \equiv \int dx \, \psi_1^*(x) \, \psi_2(x)$$

Projection of ψ_1 onto ψ_2

Orthogonal vectors $\Rightarrow \langle \psi_1 | \psi_2 \rangle = 0$

Length of vector $\|\psi\| = \sqrt{\langle \psi | \psi \rangle}$ $\equiv$ "norm"

Unit vector $\|\psi\| = 1 \Rightarrow \|\psi\|^2 = 1$

$$\|\psi\|^2 = \langle \psi | \psi \rangle = \int dx \, \psi^* \psi = 1$$

General Born Rule: Probability to find state $|\phi\rangle$ given state $|\psi\rangle$

$$P(|\phi\rangle \mid |\psi\rangle) = |\langle \phi | \psi \rangle|^2$$

Basis: Any state can be expanded in set $\{|u_n\rangle\}$

$$|\psi\rangle = \sum_n c_n |u_n\rangle$$

Orthonormal basis $\{|u_n\rangle\} \Rightarrow \langle u_n | u_m \rangle = \delta_{nm}$

$$\delta_{nm} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$ Kronecker delta

$\Rightarrow c_n = \langle u_n | \psi \rangle$: Probability Amplitude

$|c_n|^2$ = Probability to find $|u_n\rangle$

Observable: Physical quantities

Linear Operators on Hilbert space

$$\hat{A} : \mathcal{H} \to \mathcal{H}$$

$$\hat{A}(c_1|u_1\rangle + c_2|u_2\rangle) = c_1 \hat{A}|u_1\rangle + c_2 \hat{A}|u_2\rangle$$

Eigenvalue ~~eigen~~ equation: $\hat{A}|u_a\rangle = a|u_a\rangle$

eigenvalue eigenvector

When $\hat{A}$ is measured, the possible values that can be found are its eigenvalues. $\hat{A}$ physical $\Longrightarrow$ Real eigenvalue $\Rightarrow$ $\hat{A}$ Hermitian

Using Born rule, probability of finding "a"

$$P_a = |\langle u_a|\psi\rangle|^2$$

For Hermitian operator $\{|u_a\rangle\}$ = orthonormal basis

Discrete spectrum: $|\psi\rangle = \sum_a c_a|u_a\rangle$

Continuous spectrum: $|\psi\rangle = \int da\; c(a)\, |u_a\rangle$

Discrete: $\langle u_a|u_{a'}\rangle = \delta_{aa'}$ Kronecker

Continuous $\langle u_a|u_{a'}\rangle = \delta(a-a')$ Dirac

$$|\psi\rangle = \sum_a c_a|u_a\rangle \Rightarrow P_a = |\langle u_a|\psi\rangle| = |c_a|^2$$

Statistics: Given probability distribution P_a

Expectation value $= \sum_a a P_a$

Variance $\Delta a^2 = \overline{(a-\bar{a})^2} = \overline{a^2} - (\bar{a})^2$

Standard deviation $\Delta a = \sqrt{\Delta a^2}$

For Quantum state $|\psi\rangle = \sum_a c_a |u_a\rangle$

$$P_a = |c_a|^2$$

$$\Rightarrow \bar{a} = \langle \hat{A} \rangle = \int dx\, \psi^*_{(x)} \hat{A} \psi(x)$$

$$\Delta a^2 = \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2 \equiv \Delta A^2$$

Compatable Observables: $\hat{A}$ and $\hat{B}$ have a simultaneous set of eigenvectors

$$\hat{A}|u_{a,b}\rangle = a|u_{a,b}\rangle, \quad \hat{B}|u_{a,b}\rangle = b|u_{a,b}\rangle$$

$$\Longleftrightarrow \underbrace{[\hat{A}, \hat{B}]}_{\text{Commutator}} = \hat{A}\hat{B} - \hat{B}\hat{A} = 0$$

Uncertainty principle:

$$\Delta A\, \Delta B \geq \frac{1}{2} |\langle [\hat{A}, \hat{B}] \rangle|$$

E.g. $[\hat{x}, \hat{p}] = i\hbar \Rightarrow \Delta x \Delta p \geq \frac{\hbar}{2}$

Measurement: von Neumann projection postulate

If we measure observable $\hat{A}$ and find eigenvalue a, the immediately afterwards

$$|\psi\rangle \Rightarrow |u_a\rangle$$

"Collapse of the wave function"

Baysean perspective: $|\psi\rangle$ = state of knowledge

New information ⇒ reset state of knowledge

Information gain ⟺ disturbance

Compatable Observables ⇒ Can predict outcome of $\hat{A}$ and $\hat{B}$ with certainty. Possible to measure $\hat{A}$ w/o disturbing $\hat{B}$.

Ensembles: Given N identically prepared systems, measure $\hat{A}$ and each

⇒ Fraction showing eigenvalue a

$$f_a = N P_a \quad \text{in limit } N \to \infty$$

"Law of large numbers"

<u>Dynamics</u>: Given state at $t=0$ $\quad |\psi(0)\rangle$
find state at $t>0$ $\quad |\psi(t)\rangle$

State satisfies the time dependent Schrödinger

$$\frac{\partial}{\partial t}|\psi(t)\rangle = -\frac{i}{\hbar}\hat{H}\,|\psi(t)\rangle$$

$\hat{H}$ = Hamiltonian (energy observable)

Stationary state $\Rightarrow$ All probabilities constant.

$$\Rightarrow \quad |\psi(t)\rangle = e^{-iEt/\hbar}\,|u_E\rangle$$

$$\Rightarrow \quad \boxed{\hat{H}\,|u_E\rangle = E\,|u_E\rangle}$$

energy eigenstates $\rightarrow$ T. I. S. E.

Simple motion in 1D $\quad \hat{H} = \underbrace{\frac{\hat{p}^2}{2m}}_{\text{kinetic}} + \underbrace{\hat{V}(\hat{x})}_{\text{potential}}$

$$\Rightarrow \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right) u_E(x) = E\, u_E(x)$$

Differential eq.

~~Initial value problem:~~ ~~$|\psi(0)\rangle \langle \ldots \sum$~~

<u>Bound states:</u> Discrete spectrum of $\hat{H}$

<u>Unbound states:</u> Continuous spectrum of $\hat{H}$
$\longrightarrow$ Scattering

<u>Initial value problem:</u> Given $|\psi(0)\rangle$

$$|\psi(0)\rangle = \sum_E c_E \, |u_E\rangle$$

$$c_E = \langle u_E | \psi(0)\rangle$$

$$\Rightarrow |\psi(t)\rangle = \sum_E c_E \, e^{-iEt/\hbar} \, |u_E\rangle$$

Relative phases effect dynamics