Physics 492: Quantum II

Lecture 3: Foundations Continued

Outer product:

Define a linear operator $\hat{T}(\vec{U}, \vec{V})$ function of vectors

$$\hat{T}(\vec{U}, \vec{V})\vec{W} \equiv \vec{U}\,\underbrace{(\vec{V}^\dagger \cdot \vec{W})}_{\text{inner product } \vec{V} \text{ on } \vec{W}}$$

We write:
$$\hat{T}(\vec{U}, \vec{V}) = \vec{U}\vec{V}^\dagger$$
no ~~the~~ dot

$= \underline{\text{outer}}$ product $\vec{U}$ with $\vec{V}$

Matrix representation: T_{ij}

$$T_{ij} = \vec{e}_i^\dagger \cdot (\hat{T}\vec{e}_j) = (\vec{e}_i^\dagger \cdot \vec{U})(\vec{V}^\dagger \cdot \vec{e}_j)$$
$$= (\vec{e}_i^\dagger \cdot \vec{U})\,(\vec{e}_j^\dagger \cdot \vec{V})^*$$

$\Rightarrow T_{ij} = U_i V_j^*$ Outer product of two vectors = matrix

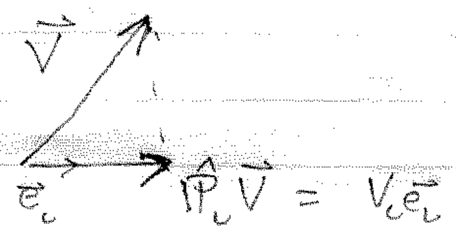
$$\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \begin{bmatrix} V_1^* & V_2^* \end{bmatrix} = \begin{bmatrix} U_1 V_1^* & U_1 V_2^* \\ U_2 V_1^* & U_2 V_2^* \end{bmatrix}$$

Example: Projection operator

Given basis $\{\vec{e}_i\}$ $\quad \vec{V} = \sum_i' V_i \, \vec{e}_i$

Define $\hat{\mathbb{P}}_i \equiv \vec{e}_i \, \vec{e}_i^{\,+}$ (outer product)

$$\Rightarrow \hat{\mathbb{P}}_i \vec{V} = \vec{e}_i \underbrace{(\vec{e}_i^{\,+} \cdot \vec{V})}_{= V_i} = V_i \vec{e}_i$$



$$\Rightarrow \vec{V} = \sum_i' (\hat{\mathbb{P}}_i \vec{V}) = \left(\sum_i' \hat{\mathbb{P}}_i\right) \vec{V}$$

$$\Rightarrow \sum_i \hat{\mathbb{P}}_i = \hat{\mathbb{1}} \quad \text{unit operator}$$

$$\boxed{\sum_i \vec{e}_i \, \vec{e}_i^{\,+} = \hat{\mathbb{1}}}$$ "Resolution of the identity"

Interpretation, the unit operator is the sum of projections on to all orthogonal directions

Matrix representation $\vec{e}_j^{\,+} \cdot \hat{\mathbb{P}}_i \vec{e}_k = (\vec{e}_j^{\,+} \cdot \vec{e}_i)(\vec{e}_i^{\,+} \cdot \vec{e}_k)$

$$= \delta_{ji} \, \delta_{ki}$$

both row and column $= i$

$$\hat{\mathbb{P}}_i = \begin{bmatrix} 0 & & & & \\ & 0 & & & 0 \\ & & \ddots 1 & & \\ 0 & & & \ddots & \\ & & & & 0 \end{bmatrix}$$

1 on the digonal for i^{th} element

$$\sum_i \hat{\mathbb{P}}_i = \begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ 0 & & & \ddots & \\ & & & & 1 \end{bmatrix} = \text{Unit matrix}$$

Adjoint Operator

Let $\hat{A}\vec{V} = \vec{W}$

for any $\vec{V}$ and mapped $\vec{W}$

Define the adjoint operator $\hat{A}^{\dagger}$ s.t.

$$\vec{W}^{\dagger} = \vec{V}^{\dagger}\hat{A}^{\dagger} \quad \text{(acts to } \underline{\text{left}}\text{)}$$

Example: Let $\hat{T} = \vec{U}\vec{V}^{\dagger}$ (outer product)

$$\Rightarrow \hat{T}^{\dagger} = \vec{V}\vec{U}^{\dagger}$$

Proof: $\hat{T}\vec{X} = \vec{U}\,(\vec{V}^{\dagger}\cdot\vec{X}) \equiv \vec{Y}$

$$\vec{Y}^{\dagger} = \vec{U}^{\dagger}\,(\vec{V}^{\dagger}\cdot\vec{X})^{*} = \vec{U}^{\dagger}(\vec{X}^{\dagger}\cdot\vec{V})$$

$$= (\vec{X}^{\dagger}\cdot\vec{V})\,\vec{U}^{\dagger} = \vec{X}^{\dagger}\cdot(\vec{V}\,\vec{U}^{\dagger})$$

$$= \vec{X}^{\dagger}\,\hat{T}^{\dagger}, \quad \hat{T}^{\dagger} = \vec{V}\,\vec{U}^{\dagger}$$

Matrix representation of adjoint operator

Let $\vec{W} = \hat{A}\,\vec{V}$

$$\Rightarrow \quad W_i = \sum_j \underbrace{A_{ij}}_{\text{matrix element}} V_j$$

$$\vec{W} \doteq \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_d \end{bmatrix} = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{1d} \\ A_{21} & A_{2} & \cdots & A_{2d} \\ \vdots & & & \\ A_{d1} & & \cdots & A_{dd} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_d \end{bmatrix}$$

$$\vec{W}^{\dagger} \doteq \begin{bmatrix} W_1^* & W_2^* & \cdots & W_d^* \end{bmatrix}$$

$$= \begin{bmatrix} V_1^* & V_2^* & \cdots & V_d^* \end{bmatrix} \begin{bmatrix} A_{11}^* & A_{21}^* & \cdots & A_{d1}^* \\ A_{12}^* & A_{22}^* & \cdots & A_{d2}^* \\ \vdots & & & \vdots \\ A_{1d}^* & & \cdots & A_{dd}^* \end{bmatrix}$$

$$= \vec{V}^{\dagger} \; \vec{A}^{\dagger}$$

$$\underset{\text{dagger}}{(A^{\dagger})_{ij}} = \Big[\underset{\text{conjugate}}{(A_{ij})^*}\Big]^{T} \leftarrow \text{transpose}$$

Dirac Notation:

Because the vector space of quantum mechanics is over complex numbers we must be careful to distinguish between vectors and their duals. This can be quite confusing and cumbersome. To deal with this Dirac introduced a new notation which all physicists use.

Vector space $\rightarrow$ kets $|\psi\rangle$

Dual space $\rightarrow$ bras $\langle\psi| = |\psi\rangle^\dagger$

Inner product: $\langle\phi|\psi\rangle$ "braket"

As before: If $|\psi\rangle = \alpha|\phi\rangle + \beta|\chi\rangle$

$$\langle\psi| = \alpha^*\langle\phi| + \beta^*\langle\chi|$$

$$\langle\phi|\psi\rangle^* = \langle\psi|\phi\rangle$$

Outer product: $|\psi\rangle\langle\phi| =$ operator $\hat{T}$

$$\hat{T}|\chi\rangle = |\psi\rangle\langle\phi|\chi\rangle$$

$$\hat{T}^\dagger = |\phi\rangle\langle\psi|$$

Dirac notation very "natural" for outer product

Adjoints: $\left(\hat{A}|\psi\rangle\right)^\dagger = \langle\psi|\hat{A}^\dagger$

Note: $(\hat{A}^\dagger)^\dagger = \hat{A}$

$$\Rightarrow \left(\hat{A}^\dagger|\psi\rangle\right)^\dagger = \langle\psi|\hat{A}$$

$$\langle\psi|\hat{A}|\phi\rangle^* = \langle\phi|\hat{A}^\dagger|\psi\rangle$$

<u>Orthonormal bases</u>: $\langle e_i|e_j\rangle = \delta_{ij}$

<u>Expansion in basis</u> $|\psi\rangle = \sum_i \;_i\, |e_i\rangle$

$$\Rightarrow \;_i = \langle e_i|\psi\rangle$$

<u>Matrix representation of operator</u>

$$A_{ij} = \langle e_i|\hat{A}|e_j\rangle$$

<u>Resolution of identity (complete set)</u>

$$\hat{\mathbb{1}} = \sum_i |e_i\rangle\langle e_i| \quad \text{(sum over projections)}$$

Use for representation:

$$|\psi\rangle = \hat{\mathbb{1}}|\psi\rangle = \sum_i |e_i\rangle\underbrace{\langle e_i|\psi\rangle}_{\psi_i}$$

$$\Rightarrow |\psi\rangle = \sum_i \psi_i\, |e_i\rangle$$

$$\hat{A} = \hat{\mathbb{1}}\,\hat{A}\,\hat{\mathbb{1}} = \left(\sum_i |e_i\rangle\langle e_i|\right)\hat{A}\left(\sum_j |e_j\rangle\langle e_j|\right)$$

$$= \sum_{ij} |e_i\rangle\langle e_i|\hat{A}|e_j\rangle\langle e_j|$$

$$\Rightarrow \boxed{\hat{A} = \sum_{ij} A_{ij}\,|e_i\rangle\langle e_j|}$$ Sum of "transition operators"

<u>Using Dirac Notation</u> (Examples)

- $\langle\phi|\psi\rangle = \sum_i \langle\phi|e_i\rangle\langle e_i|\psi\rangle = \underbrace{\sum_i \phi_i^* \psi_i}_{\text{coordinate rep of inner product}}$

  $\uparrow$ insert complete set

- Let $|\phi\rangle = \hat{A}|\psi\rangle \Rightarrow \phi_i = \langle e_i|\hat{A}|\psi\rangle$

  $\uparrow$ insert complete set

$$\Rightarrow \phi_i = \sum_j \langle e_i|\hat{A}|e_j\rangle\langle e_j|\psi\rangle = \underbrace{\sum_j A_{ij}\psi_j}_{\text{matrix multiplication}}$$

- Prove $(A^\dagger)_{ij} = (A^*_{ij})^T = A^*_{ji}$

$$(A^\dagger)_{ij} = \langle e_i|\hat{A}^\dagger|e_j\rangle = \langle e_j|\hat{A}|e_i\rangle^* = A^*_{ji}$$