Physics 492: Quantum Mechanics II

Lecture 6: Symmetries and Degeneracies – Introduction to Angular Momentum

2D Harmonic oscillator Consider a particle in a 2D harmonic well:

$$V(x,y) = \frac{1}{2}m\omega_x^2 x^2 + \frac{1}{2}m\omega_y^2 y^2, \quad \text{where} \quad \omega_x = \sqrt{\frac{K_x}{m}}, \quad \omega_y = \sqrt{\frac{K_y}{m}}$$

$\leftarrow$ spring constant along-x, $\leftarrow y$

$\Rightarrow$ Separable! $\hat{H} = \hat{H}^{(x)} + \hat{H}^{(y)} = \left(\frac{\hat{p}_x^2}{2m} + \frac{1}{2}m\omega_x^2 x^2\right) + \left(\frac{\hat{p}_y^2}{2m} + \frac{1}{2}m\omega_y^2 y^2\right)$

$\Rightarrow$ Energy eigenvectors: $|n_x, n_y\rangle \quad \{n_x, n_y = 0, 1, 2, 3 \ldots\}$

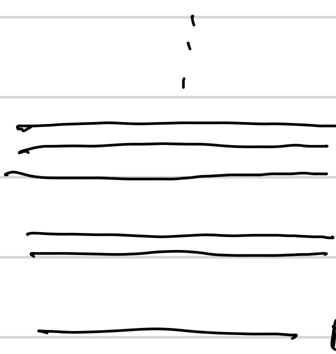
Energy eigenfunctions: $\langle x, y | n_x, n_y\rangle = \langle x|n_x\rangle\langle y|n_y\rangle = u_{n_x}(x)\, u_{n_y}(y)$

$$\psi_{n_x n_y}(x,y) = \underbrace{H_{n_x}\left(\sqrt{2}\frac{x}{x_c}\right) H_{n_y}\left(\sqrt{2}\frac{y}{y_c}\right) e^{-\frac{x^2}{x_c^2}}\, e^{-\frac{y^2}{y_c^2}}}_{\text{Hermite Gaussian}} \qquad x_c = \sqrt{\frac{2\hbar}{m\omega_x}}, \quad y_c = \sqrt{\frac{2\hbar}{m\omega_y}}$$

$$\hat{H}|n_x, n_y\rangle = (E_{n_x} + E_{n_y})|n_x, n_y\rangle = \hbar\omega_x(n_x + \tfrac{1}{2}) + \hbar\omega_y(n_y + \tfrac{1}{2})$$

Suppose the spring along x is "stiffer" than that along y, eg $\omega_x = \frac{5}{4}\omega_y$

Energy level diagram:



(2,0) $E_{2,0} = 2\hbar\omega_x + E_0 = \frac{5}{2}\hbar\omega_y + E_0$

(1,1) $E_{1,1} = \hbar\omega_x + \hbar\omega_y + E_0 = \frac{9}{4}\hbar\omega_y + E_0$

(0,2) $E_{0,2} = 2\hbar\omega_y + E_0$

(1,0) $E_{1,0} = \hbar\omega_x + E_0 = \frac{5}{4}\hbar\omega_y + E_0$

(0,1) $E_{0,1} = \hbar\omega_y + E_0$

(0,0) $E_{00} = \frac{\hbar\omega_x}{2} + \frac{\hbar\omega_y}{2}$

Suppose $\omega_x = \omega_y \equiv \omega \qquad \hat{H} = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}m\omega^2(x^2+y^2)$

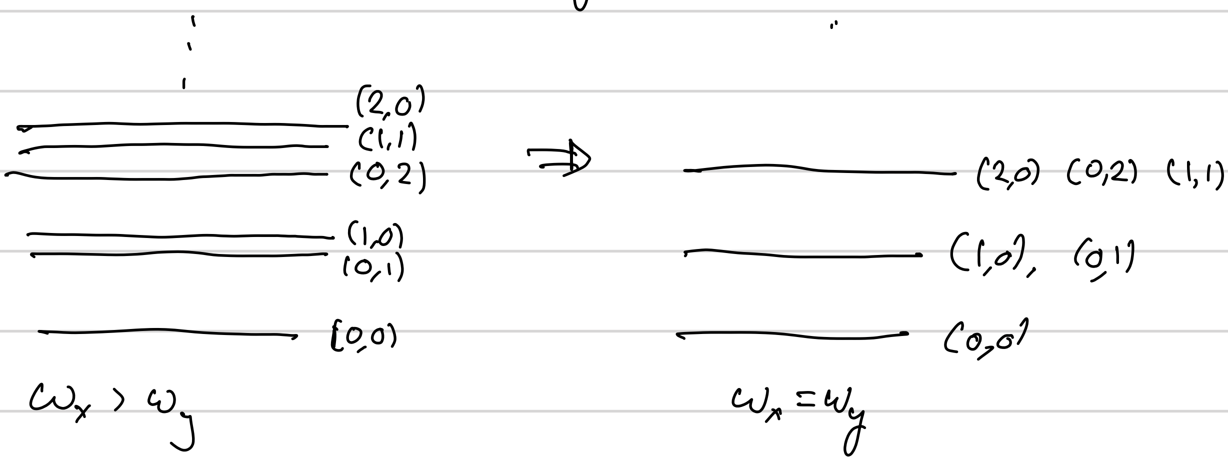
Isotropic SHO in 2D

$$\psi_{n_x, n_y} = H_{n_x}\left(\frac{x}{l_c}\right) H_{n_y}\left(\frac{y}{l_c}\right) e^{-\frac{x^2+y^2}{l_c}} \qquad l_c \equiv \sqrt{\frac{2\hbar\omega}{m}}$$

$$E_{n_x, n_y} = \hbar\omega(n_x + n_y + 1)$$

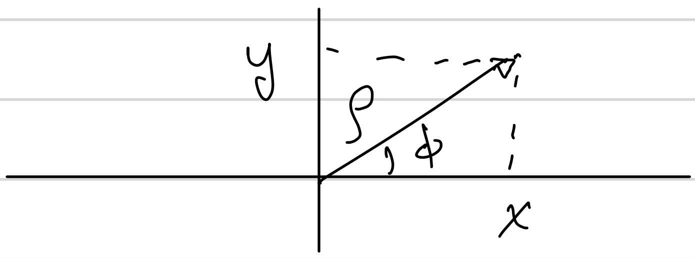
Here there is symmetry $x \leftrightarrow y \Rightarrow$ degeneracy

But wait, here the degeneracy is different!



$$\boxed{\begin{array}{l}\text{Energy levels are now } E_{n_x,n_y} = \hbar\omega(\underbrace{n_x + n_y}_{n} + 1) = \hbar\omega(n+1) \quad \text{energy depend on sum} \\ \text{Degeneracy: } g_n = \binom{n+2-1}{n} = n-1\end{array}}$$

Cylindrical Coordinates



$x = \rho\cos\phi \qquad \rho = \sqrt{x^2+y^2}$

$y = \rho\sin\phi \qquad \tan\phi = y/x$

$\Rightarrow V(x,y) = \frac{1}{2}m\omega^2\rho^2 \Leftarrow$ Potential independent of ϕ

<u>azimuthal symmetry</u>

$\Rightarrow$ Force only radial: $\frac{\partial V}{\partial \rho}$. No tangental force

this is an example of a <u>central potential</u>

Because the force is directed radially there is no torque on the particle

$\Rightarrow$ In a central potential <u>angular momentum</u> is conserved

<u>Angular momentum</u>: $\vec{L} = \vec{r} \times \vec{p}$ (a very important physical quantity)

For motion in the x-y plane we have angular momentum along the <u>z-axis</u>: $L_z = x p_y - y p_z$

Quantum: $\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_z \doteq \frac{\hbar}{i} x \frac{\partial}{\partial y} - \frac{\hbar}{i} y \frac{\partial}{\partial x}$ (in position representation)

Changing to cyclindrical coordinates: $\frac{\partial}{\partial \phi} = \frac{\partial x}{\partial \phi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \phi}\frac{\partial}{\partial y} = (-\rho\sin\phi)\frac{\partial}{\partial x} + (\rho\cos\phi)\frac{\partial}{\partial y}$

$= x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$

$\Rightarrow$ In position representation in clindrical coordinate, $\boxed{\hat{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}}$

Now when $V(\hat{\rho})$ $[\hat{L}_z, V(\hat{\rho})] = 0 \Rightarrow$ Angular momentum commute with a general potential. As we can also show
$[\hat{L}_z, \hat{p}_x^2 + \hat{p}_y^2] = 0 \Rightarrow [\hat{L}_z, \hat{H}] = 0$ for a central potential.

By Ehrenfest's theorem: $\frac{d}{dt}\langle \hat{L}_z \rangle = \frac{i}{\hbar}\langle [\hat{L}_z, \hat{H}] \rangle = 0$

$\Rightarrow$ <u>Angular momentum is conserved.</u> (no torque)

Said another way, consider the operator $\hat{O}(\phi_0) = e^{-i\phi_0 \hat{L}_z/\hbar} \doteq e^{-\phi_0 \frac{\partial}{\partial\phi}}$

$$e^{-\phi_0 \frac{\partial}{\partial\phi}} f(\phi) = f(\phi - \phi_0)$$: Translation in ϕ_0 $\Rightarrow$ <u>Rotation</u>

Linear momentum p_x:	Generator of displacement along x – position translation
Angular momentum $\hat{L}_z$:	Generator of displacement along ϕ – rotation

- "Flat" potential along x, i.e. $\hat{V}$ translational invariant

  $\Rightarrow [\hat{H}, \hat{p}_x] = 0 \Rightarrow$ <u>momentum conserved</u> $\frac{d}{dt}\langle \hat{p}_x \rangle = 0$

  $\Rightarrow \exists$ common eigenstates of $\hat{H}$ and $\hat{p}_x$: Plane waves

- "Axial symmetric" potential, i.e. $\hat{V}$ rotationally invariant around z-axis.

  $\Rightarrow [\hat{H}, \hat{L}_z] = 0 \Rightarrow$ <u>angular momentum conserved</u> $\frac{d}{dt}\langle \hat{L}_z \rangle = 0$

  $\Rightarrow \exists$ common eigenstates of $\hat{H}$ and $\hat{L}_z$.

Eigenstate of $\hat{L}_z$: $\hat{L}_z |\lambda\rangle = \lambda |\lambda\rangle$, $\Rightarrow \frac{\hbar}{i}\frac{\partial}{\partial\phi}\Phi_\lambda(\phi) = \lambda\, \Phi_\lambda(\phi)$ (as wave function)

$\Rightarrow \Phi_\lambda(\phi) = A e^{i\frac{\lambda}{\hbar}\phi}$, normalized $\int_0^{2\pi} |\Phi_\lambda(\phi)|^2 = 1 \Rightarrow A = \frac{1}{\sqrt{2\pi}}$

Eigenfunction of $\hat{L}_z$ $\quad \Phi_\lambda(\phi) = \frac{1}{\sqrt{2\pi}} e^{\frac{i\lambda}{\hbar}\phi}$

But, the wave function must be single valued: $\Phi_\lambda(\phi+2\pi) = \Phi_\lambda(\phi)$

$$\Rightarrow \quad e^{i\frac{2\pi\lambda}{\hbar}} = 1 \quad \Rightarrow \frac{\lambda}{\hbar} = m \; : \; \underline{\text{integer}}$$

$$\Rightarrow \boxed{\text{Eigenfunctions of } \hat{L}_z \; \Phi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi} \qquad m = 0, \pm 1, \pm 2, \ldots}$$

Note: Here the "quantization" is due to the "compactness" of angles

Let us return to 2D isotropic SHO.

Consider the doubly-degenerate first excited state $n=1$ ($E_1 = \hbar\omega + E_0$)

We can two wave functions that we have written

$$|n_x=1, n_y=0\rangle \doteq \psi_{n_x=1, n_y=0}(x,y) = H_1\left(\frac{x}{s}\right) H_0\left(\frac{y}{s}\right) e^{-\frac{x^2+y^2}{s^2}} \propto x\, e^{-\frac{x^2+y^2}{s^2}}$$

$$|n_x=0, n_y=1\rangle \doteq \psi_{n_x=0, n_y=1}(x,y) = H_0\left(\frac{x}{s}\right) H_1\left(\frac{y}{s}\right) e^{-\frac{x^2+y^2}{s^2}} \propto y\, e^{-\frac{x^2+y^2}{s^2}}$$

Because these are degenerate in energy, and superposition of these two states is an energy eigenstate with the same energy.

Consider $|1, \pm\rangle \equiv \frac{|n_x=1, n_y=0\rangle \pm i |n_x=0, n_y=1\rangle}{\sqrt{2}} \propto (x \pm iy) e^{-\frac{x^2+y^2}{s^2}}$

$$\Rightarrow |1, \pm 1\rangle \doteq \rho e^{-\frac{\rho^2}{s^2}} e^{\pm i\phi} \qquad \text{(in cylindrical coordinates)}$$

these states are eigenstates of $\hat{L}_z$ with eigenvalue $m = \pm 1$

The fact that $\exists$ eigenstates of $\hat{H}$ that are also eigenstates of $\hat{L}_z$ is not a surprise, $[\hat{H}, \hat{L}_z] = 0$.

Note, because of degeneracy, the energy eigenvalue alone does not specify the eigenstate

Complete Set of Commuting Operators: (CSCO)

For system with N degrees of freedom (e.g. $N=2$ for 2D SHO) we can completely specify the state as the common eigenstates of N multual commuting operators.

Example: $\{\hat{H}^{(x)}, \hat{H}^{(y)}\}$ form a CSCO.

Joint eigenstate $|n_x, n_y\rangle$, $\hat{H}|n_x, n_y\rangle = (n_x + n_y + 1)\hbar\omega\,|n_x, n_y\rangle$

Degeneracy: $n_x + n_y = n$ $\quad g_n = n-1$

Wave function: $\psi_{n_x n_y}(x,y) = u_{n_x}(x)\,u_{n_y}(y) = H_{n_x}\left(\frac{x}{x_c}\right) H_{n_y}\left(\frac{y}{y_c}\right) e^{-\frac{x^2}{x_c^2}} e^{-\frac{y^2}{y_c^2}}$

$\{\hat{H}, \hat{L}_z\}$ form a CSCO.

Joint eigenstate $|n, m\rangle$: $\hat{H}|n,m\rangle = (n+1)\hbar\omega\,|n,m\rangle$

$\hat{L}_z|n,m\rangle = m\hbar\,|n,m\rangle$

$g_n = n-1 \quad \{m = -n, -n+2, \dots n-2, n\}$

Wave function $\psi_{n,m}(\rho,\phi) = R_{n,m}(\rho)\,\Phi_m(\phi) = \underbrace{R_{n,m}(\rho)}_{\text{Radial wave function}} e^{im\phi}$

Symmetries, degeneracies, conserved quantities

The isotropic 2D SHO is rotationally symmetric around z-axis.

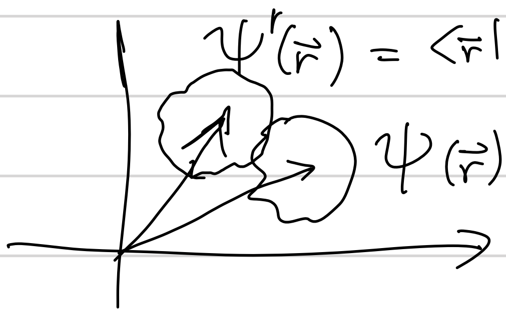
Because of this, there is no torque (central force) and the z-component of angular momentum is conserved. This relationship between symmetry in the Hamiltonian and conserved physical quantities is no accident. It is an example of Noether's theorem, first established in classical mechanics. Recall in classical mechanics, for a system with N-degrees of freedom, when there are N constants the system is said to be integrable (regular motion). For example, for 2D isotropic SHO, (E, L_z) specify the trajectory.

Quantum mechanically, we also see the relationship between symmetry of the Hamiltonian and "constants of the motion". Because of the azimuthal symmetry of the Hamiltonian, $[\hat{H}, \hat{L}_z] = 0 \Rightarrow \hat{L}_z$ is conserved.
A physical symmetry in quantum mechanics is implemented by a unitary map in Hilbert space. In particular the "generator" of the transformation in the conserved quantity, i.e. $\hat{L}_z$ is the "generator" of rotations around the z-axis

$$\hat{U}(\phi) \equiv e^{-i\phi \hat{L}_z/\hbar} \doteq e^{-\phi \frac{\partial}{\partial\phi}} \quad \text{is position space}$$

$$\hat{U}(\phi) |\vec{r}\rangle = |\mathbb{R}(\phi)\vec{r}\rangle \qquad \mathbb{R}(\phi) = \text{rotation map on position}$$

$$\hat{U}(\phi)|\psi\rangle = \text{rotated state} = |\psi'\rangle \qquad \psi'(\vec{r}) = \langle \vec{r}|\hat{U}(\phi)|\psi\rangle = \psi(\mathbb{R}^{-1}(\phi)\vec{r})$$



Symmetry and degeneracy:
Consider an energy eigenvector: $\hat{H}|u_E\rangle = E|u_E\rangle$
Suppose $\hat{H}$ is invariant under some symmetry $\hat{U}^\dagger \hat{H} \hat{U} = \hat{H}$
$\Rightarrow [\hat{H}, \hat{U}] = 0$
Let $\hat{U}|u_E\rangle = |u'_E\rangle$
Suppose $|u'_E\rangle \neq |u_E\rangle \Rightarrow \hat{H}|u'_E\rangle = \hat{H}\hat{U}|u_E\rangle = \hat{U}\hat{H}|u_E\rangle = E\hat{U}|u_E\rangle = E|u'_E\rangle$
$\Rightarrow$ $|u'_E\rangle$ and $|u_E\rangle$ are degenerate

If $\hat{U}$ is "continuous symmetry", e.g. translation along line or angle
$\Rightarrow e^{i\lambda\hat{G}} \qquad \hat{G} = \text{generator}$
$\Rightarrow [\hat{H}, \hat{G}] = 0 \qquad \hat{G}$ is constant of motion

$\Rightarrow \exists$ common eigenstates of $\{\hat{H}, \hat{G}\}$
For N d.o.f. $\{\hat{H}, \hat{G}_1, \hat{G}_2, \ldots \hat{G}_{N-1}\}$ is a CSCO with Energy and $G_1\ G_2 \ldots G_N$ "constants of motion"