Physics 492: Quantum Mechanics II
Lecture 7 — Spherical Symmetry and Angular Momentum

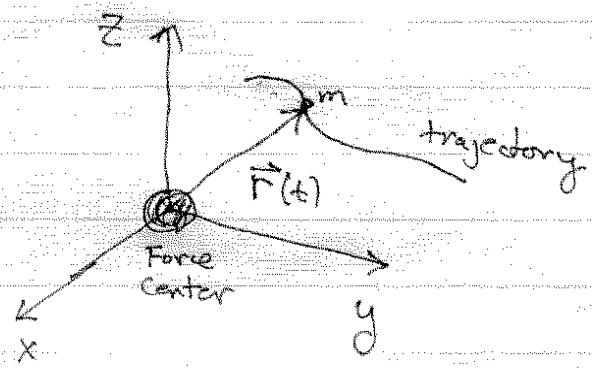
A particularly important class of potentials in physics are "central potentials" which depend only on the distance of the particle to the origin.

$$\Rightarrow V(x,y,z) = V(r)$$

$$\text{where } r = \sqrt{x^2+y^2+z^2}$$

Examples are the gravitation potential and Coulomb potential with one particle at the origin and another a distance r away. The latter is particularly important in quantum mechanics where electromagnetic forces dominate the properties of atoms, molecules, and condensed matter.

Classical Mechanics of Central Potentials



The equipotential surfaces are spheres: $V(r) = \text{constant}$

Force: $\vec{F} = -\vec{\nabla} V$

$$\Rightarrow \vec{F} = -\frac{\partial V(r)}{\partial r}\,\vec{e}_r$$

(central force)

where $\vec{e}_r = \dfrac{\vec{r}}{|\vec{r}|} = \dfrac{\vec{r}}{r}$

An important physical quantity is <u>angular momentum</u>

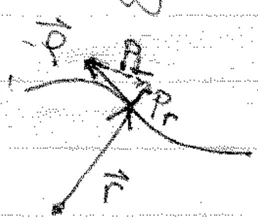
$$\vec{L} = \vec{r} \times \vec{P} \quad \text{where } \vec{p} = m\dot{\vec{r}} = \text{momentum}$$

In a central potential $\vec{L}$ is conserved

since $\frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times \vec{F} = \vec{r} \times \vec{e}_r F(r) = 0$

$\qquad\qquad\qquad\qquad\qquad\;\; \text{torque}$

Kinetic energy: $T = \frac{p^2}{2m} = \frac{|\vec{p}|^2}{2m}$



We can break up the momentum into a radial component and a perpendicular component.

$p_r \equiv \vec{e}_r \cdot \vec{p}$ (radial momentum)

$\vec{p}_\perp \equiv \vec{p} - p_r \vec{e}_r \quad , \quad |\vec{p}|^2 = p_r^2 + p_\perp^2$

Now $|\vec{L}| = |\vec{r} \times \vec{p}| = r\, p_\perp \Rightarrow p_\perp = \frac{L}{r}$

$\Rightarrow$ Perpendicular momentum $\propto$ Angular momentum

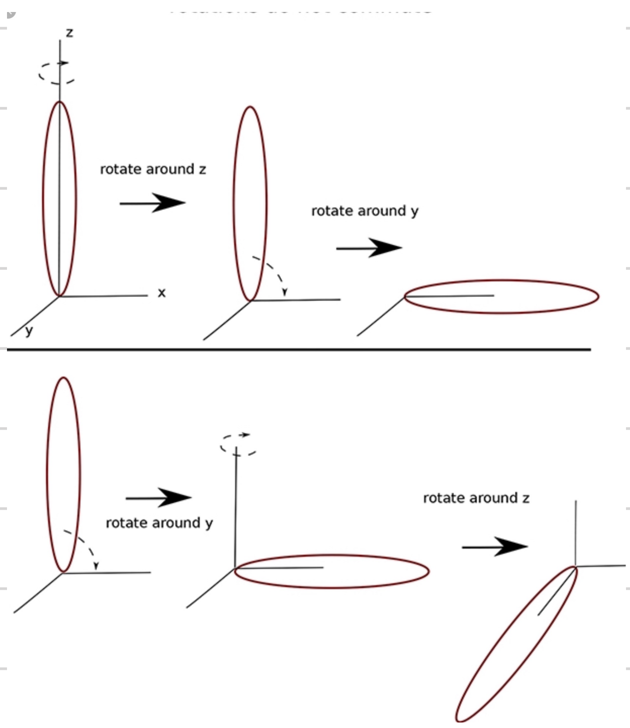
$$\boxed{\frac{|\vec{p}|^2}{2m} = \frac{p_r^2}{2m} + \frac{L^2}{2mr^2}}$$

Complete set of commuting operators

Given a spherical symmetric potential $V(r)$, the Hamiltonian is invariant about a rotation of <u>any axis</u>. This means $\hat{H}$ commutes with <u>all</u> components of angular momentum

$$[\hat{H}, \hat{L}_x] = [\hat{H}, \hat{L}_y] = [\hat{H}, \hat{L}_z] = 0$$

So it seems we might choose $\hat{H}$ and and two components of $\vec{L}$ e.g. $\{\hat{H}, \hat{L}_x, \hat{L}_z\}$ as a C.S.C.O to completely definite the energy eigenstates. But wait, something odd happens when considering rotations around different axes:



Rotation around different axes do not <u>Commute</u>. This is a fact of <u>geometry</u>. This is then reflected in the fact the the different "generators of rotation" the components of angular momentum do not commute.

To see this formally, consider the components of $\hat{\vec{L}}$

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y, \quad \hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z, \quad \hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$$

$$[\hat{L}_x, \hat{L}_y] = [\hat{y}\hat{p}_z - \hat{z}\hat{p}_y, \hat{z}\hat{p}_x - \hat{x}\hat{p}_z] = [\hat{y}\hat{p}_z, \hat{z}\hat{p}_x] + [\hat{z}\hat{p}_y, \hat{x}\hat{p}_z]$$

$$= \hat{y}[\hat{p}_z, \hat{z}]\hat{p}_x + \hat{x}[\hat{z}, \hat{p}_z]\hat{p}_y = i\hbar(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x) = i\hbar\hat{L}_z$$

$$\Rightarrow \boxed{[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z}$$

More generally, if we index the components $i = 1,2,3 \Rightarrow x, y, z$

Then $\boxed{[\hat{L}_i, \hat{L}_j] = i\hbar\, \epsilon_{ijk}\, \hat{L}_k}$ $\quad \epsilon_{ijk}$ = Levi-Chivita anti-symmetric tensor

Thus, unlike for linear momentum $\hat{\vec{p}}$, with eigenvectors $|\vec{p}\rangle = |p_x, p_y, p_z\rangle$ that are simultaneous eigenvectors of $\hat{p}_x$, $\hat{p}_y$, and $\hat{p}_z$. There is no eigenvector $|\vec{L}\rangle = |L_x, L_y, L_z\rangle$ since the components of $\vec{L}$ don't commute

Angular momentum uncertainty relation.

Components of angular momentum don't commute we have a generalized Heisenberg uncertainty principle

$$\boxed{\Delta L_x \Delta L_y \geq \frac{\hbar}{2} |\langle [\hat{L}_x, \hat{L}_y] \rangle| = \frac{\hbar}{2} \langle \hat{L}_z \rangle}$$

Consider however $\hat{L}^2 = \hat{\vec{L}} \cdot \hat{\vec{L}} = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$ Claim: $\boxed{[\hat{L}^2, \hat{L}_z] = 0}$

Check $[\hat{L}^2, \hat{L}_z] = [\hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2, \hat{L}_z] = [\hat{L}_x^2, \hat{L}_z] + [\hat{L}_y^2, \hat{L}_z]$

$$= \hat{L}_x [\hat{L}_x, \hat{L}_z] + [\hat{L}_x, \hat{L}_z]\hat{L}_x + \hat{L}_y [\hat{L}_y, \hat{L}_z] + [\hat{L}_y, \hat{L}_z]\hat{L}_y$$

$$= i\hbar \left(-\hat{L}_x \hat{L}_y - \hat{L}_y \hat{L}_x + \hat{L}_y \hat{L}_x + \hat{L}_x \hat{L}_y \right) = 0 \quad \checkmark$$

It is each to show that in general, $[\hat{L}^2, \hat{L}_i]$ for and component $\hat{L}_i$

Thus, the best we can do in specifying eigenvectors of angular momentum is to find simultaneous eigenvectors of $\hat{L}^2$ and one component of $\hat{\vec{L}}$

"Standard basis" Simultaneous eigenvectors of $\hat{L}^2$ and $\hat{L}_z$: $|l, m\rangle$

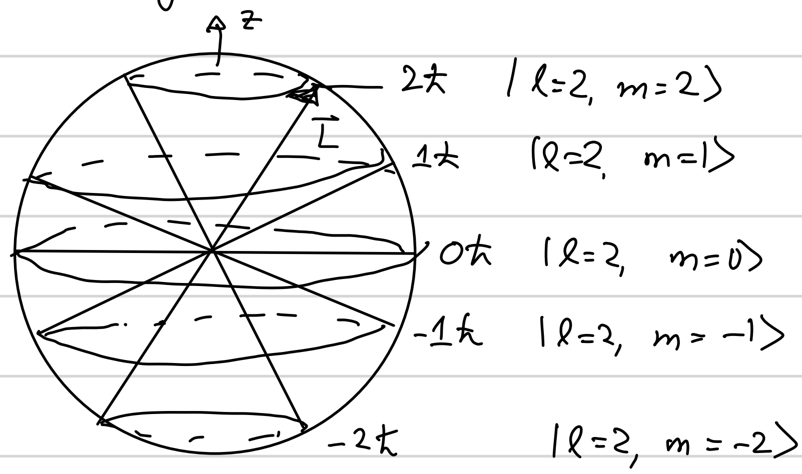
We have already seen $\hat{L}_z |l, m\rangle = m\hbar\, |l, m\rangle$ $\quad m$ = integer

We state here (and prove later)

$\hat{L}^2 |l, m\rangle = l(l+1)\hbar^2 |l, m\rangle$ $\quad l = 0, 1, 2, \ldots$ (non-neg integer)

for a given l, $l \leq m \leq -l$, $\quad m = -l, -l+1, \ldots -1, 0, +1, \ldots l$

We can graphically depict the eigenvectors $|\ell, m\rangle$ on a sphere radius $\sqrt{\ell(\ell+1)}\,\hbar$. Eg. $\ell = 2$



This vector picture denote that the projection of $\vec{L}$ along the z-axis is fix at $m\hbar$ where $-2 \le m \le 2$ in integer step. The length of the vector is ~~also~~ fixed at $\sqrt{2(2+1)}\,\hbar = \sqrt{6}\,\hbar \approx 2.45\,\hbar$. The x and y components of $\vec{L}$ are not definite, and can lie anywhere on the circle. One can show (see homework) that in the standard bases

$$\Delta L_x = \Delta L_z = \hbar\sqrt{\frac{\ell(\ell+1) - m^2}{2}}$$

But according to the uncertainty principle $\Delta L_x \Delta L_y \ge \frac{\hbar}{2}|\langle L_z\rangle| = |m|\frac{\hbar}{2}$

$\Rightarrow$ Minimal uncertainty state $m = \pm \ell$ "stretched state"

$$\Delta L_x = \Delta L_y = \sqrt{\frac{\ell}{2}}\,\hbar$$

<u>Algebraic Solution to angular momentum eigenvalues</u>

We now seek to derive the eigenvalues of $\hat{L}^2$ and their relation to $\hat{L}_z$. There are two approaches – Solve differential equations in the position representation, or use operator algebra. We will take the latter approach here. We will be guided by our solution of the energy eigenvalues of the SHO.

Recall in 1D SHO: $\hat{H} = \hbar\omega(\hat{X}^2 + \hat{P}^2)$.

We "factorized" $\hat{X}^2 + \hat{P}^2 = \frac{(\hat{X} + i\hat{P})(\hat{X} - i\hat{P}) + (\hat{X} - i\hat{P})(\hat{X} + i\hat{P})}{2} = \frac{\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger}{2}$

Where $\hat{a} = \hat{X} + i\hat{P}$ and $\hat{a}^\dagger = \hat{X} - i\hat{P}$ are the <u>ladder operators</u>

$$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle \qquad \hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

We seek the solution the the joint eigenvector of $\hat{L}^2$ and $\hat{L}_z$
Let us first remove the units and define dimensionless operators $\hat{\vec{\ell}} \equiv \frac{\hat{\vec{L}}}{\hbar}$

$$[\hat{\ell}_i, \hat{\ell}_j] = i\epsilon_{ijk}\hat{\ell}_k$$

We seek the common eigenvectors of $\hat{\ell}^2 = \hat{\vec{\ell}}\cdot\hat{\vec{\ell}}$ and $\hat{\ell}_z$. Let us define

$$\hat{\ell}^2|\lambda, m\rangle = \lambda^2|\lambda, m\rangle, \quad \hat{\ell}_z|\lambda, m\rangle = m|\lambda, m\rangle$$

$$\lambda \geq 0 \qquad\qquad m = \text{integer}$$

$$\hat{\ell}^2 = \hat{\ell}_x^2 + \hat{\ell}_y^2 + \hat{\ell}_z^2 \Rightarrow \hat{\ell}_x^2 + \hat{\ell}_y^2 = \hat{\ell}^2 - \hat{\ell}_z^2 : \quad (\hat{\ell}_x^2 + \hat{\ell}_y^2)|\lambda, m\rangle = (\lambda^2 - m^2)|\lambda, m\rangle$$

Now $\hat{\ell}_x^2 + \hat{\ell}_y^2$ is a "positive operator" $\Rightarrow \langle \hat{\ell}_x^2 + \hat{\ell}_y^2\rangle \geq 0$ for any state
In particular $\langle\lambda, m|\hat{\ell}_x^2 + \hat{\ell}_y^2|\lambda, m\rangle = \lambda^2 - m^2 \geq 0 \Rightarrow -\lambda \leq m \leq \lambda$

Now, motivated by the algebraic structure of the SHO we define

$$\boxed{\hat{\ell}_+ \equiv \hat{\ell}_x + i\hat{\ell}_y \quad , \quad \hat{\ell}_- = \hat{\ell}_x - i\hat{\ell}_y}$$

Note $\hat{\ell}_+^\dagger = \hat{\ell}_-$, $\hat{\ell}_-^\dagger = \hat{\ell}_+$

$$\hat{\ell}_+\hat{\ell}_- = \hat{\ell}_x^2 + \hat{\ell}_y^2 - i(\hat{\ell}_x\hat{\ell}_y - \hat{\ell}_y\hat{\ell}_y) = \hat{\ell}^2 - \hat{\ell}_z^2 + \hat{\ell}_z = \hat{\ell}^2 - \hat{\ell}_z(\hat{\ell}_z - 1)$$

$$\hat{\ell}_-\hat{\ell}_+ = \hat{\ell}_x^2 + \hat{\ell}_y^2 + i(\hat{\ell}_x\hat{\ell}_y - \hat{\ell}_y\hat{\ell}_y) = \hat{\ell}^2 - \hat{\ell}_z^2 - \hat{\ell}_z = \hat{\ell}^2 - \hat{\ell}_z(\hat{\ell}_z + 1)$$

$$\Rightarrow \hat{\ell}_x^2 + \hat{\ell}_y^2 = \frac{\hat{\ell}_+\hat{\ell}_- + \hat{\ell}_-\hat{\ell}_+}{2} \qquad \left(\text{like } \hat{X}^2 + \hat{P}^2 = \frac{\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger}{2}\right)$$

We will show that these operators are the <u>ladder operators</u> of angular momentum
Note the combination $x \pm iy$ is important in rotations about the z-axis, as we have seen.
The new components $\{\hat{\ell}_+, \hat{\ell}_-, \hat{\ell}_z\}$ are now the important players. We have commutations:

$$\boxed{\begin{aligned} [\hat{\ell}_+, \hat{\ell}_-] &= [\hat{\ell}_x + i\hat{\ell}_y, \hat{\ell}_x - i\hat{\ell}_y] = 2\hat{\ell}_z \\ [\hat{\ell}_z, \hat{\ell}_\pm] &= [\hat{\ell}_z, \hat{\ell}_x \pm i\hat{\ell}_y] = \pm\hat{\ell}_\pm \\ [\hat{\ell}^2, \hat{\ell}_\pm] &= [\hat{\ell}^2, \hat{\ell}_z] = 0 \end{aligned}}$$

With these commutation relations in hand, we can solve for the eigenvalues.

Lemma: $\hat{l}_\pm |\lambda, m\rangle = c_\pm |\lambda, m\pm 1\rangle$. That is, $\hat{l}_\pm|\lambda, m\rangle$ is an eigenvector of $\hat{l}^2$ with eigenvalue λ^2 and $\hat{l}_z$ with eigenvalue $m\pm 1$. $\hat{l}_+$ "raises" the m quantum plus $+1$ and $\hat{l}_-$ lowers it by -1 with changing the $\hat{l}^2$ eigenvalue

Proof: $\hat{l}^2(\hat{l}_\pm|\lambda,m\rangle) = \hat{l}_\pm \hat{l}^2|\lambda,m\rangle = \lambda^2(\hat{l}_\pm|\lambda,m\rangle)$ since $\hat{l}^2$ and $\hat{l}_\pm$ commute.

$$\hat{l}_z(\hat{l}_\pm|\lambda,m\rangle) = (\hat{l}_\pm\hat{l}_z + [\hat{l}_z, \hat{l}_\pm])|\lambda,m\rangle = (\hat{l}_\pm \hat{l}_z \pm \hat{l}_\pm)|\lambda,m\rangle = (m\pm 1)|\lambda,m\rangle$$

Lemma: There exists an m_{max} such that $\hat{l}_+|\lambda, m_{max}\rangle = 0$ and m_{min} such that $\hat{l}_-|\lambda, m_{min}\rangle = 0$

Proof: We showed that $-\lambda \le m \le \lambda$ where $\lambda > 0$. Now, there must be a maximum value for $m \equiv m_{max}$ such that $\hat{l}_+|\lambda, m_{max}\rangle = 0$. Otherwise $\exists$ some m such that $m+1 > \lambda$ and that state is not possible. The "ladder" must have a "highest rung", or we will raise m above λ. Similarly, there must exist a minimum value of m, such that $\hat{l}_-|\lambda, m_{min}\rangle = 0$. Otherwise, for some $m<0$ $m-1 < \lambda$, and that state is impossible. The ladder must have a "lowest rung"

Lemma $m_{max} = -m_{min} \equiv l$, $l = \text{integer} \ge 0$

$$\lambda = l(l+1)$$

By definition $\hat{l}_+|\lambda, m_{max}\rangle = 0 \Rightarrow \langle\lambda, m_{max}|\hat{l}_-\hat{l}_+|\lambda, m_{max}\rangle = 0$

$$\Rightarrow \langle\lambda, m_{max}|\hat{l}^2 - \hat{l}_z(\hat{l}_z+1)|\lambda, m_{max}\rangle = \lambda^2 - m_{max}(m_{max}+1) = 0$$

$$\Rightarrow m_{max}(m_{max}+1) = \lambda^2$$

Similarly $\hat{l}_-|\lambda, m_{min}\rangle = 0 \Rightarrow \langle\lambda, m_{min}|\hat{l}_+\hat{l}_-|\lambda, m_{min}\rangle = 0$

$$\Rightarrow \langle\lambda, m_{min}|\hat{l}^2 - \hat{l}_z(\hat{l}_z-1)|\lambda, m_{min}\rangle = \lambda^2 - m_{min}(m_{min}-1) = 0$$

$$\Rightarrow m_{min}(m_{min}-1) = \lambda^2$$

$$\Rightarrow m_{max}(m_{max}+1) = m_{min}(m_{min}-1) \Rightarrow m_{max} = -m_{min}$$

And, we previously showed using position representation, eigenvalues of $\hat{L}_z$: $m\hbar$ with m integer $\Rightarrow m_{max} = -m_{min} = \text{integer} \equiv l$ ✓

To summarize, we have shown that, there exist common eigenstates of $\hat{l}^2$ and $\hat{l}_z$ with eigenvalues labled l and m, such that

$$\hat{l}^2|l,m\rangle = l(l+1)\,|l,m\rangle \qquad \underline{l \text{ integer}} \geq 0$$

$$\hat{l}_z|l,m\rangle = m|l,m\rangle, \qquad \underline{m \text{ is an integer}} \quad -l \leq m \leq l :$$

For a given l, $m = -l, -l+1, \dots -1, 0, +1, \dots, l-1, l$: $2l+1$ different m's.

$$\hat{l}_\pm |l,m\rangle = c_\pm\, |l, m\pm 1\rangle, \quad \text{with } \hat{l}_\pm|l,\pm l\rangle = 0$$

To find the normalization constant $c_\pm$, $\| \hat{l}_\pm |l,m\rangle \|^2 = \langle l,m| \hat{l}_\pm^\dagger \hat{l}_\pm |l m\rangle = |c_\pm|^2$

$$\Rightarrow |c_\pm|^2 = \langle l,m|\, \hat{l}_\mp \hat{l}_\pm\, |l,m\rangle = \underline{\langle l,m|\, \hat{l}^2 - \hat{l}_z(\hat{l}_z \pm 1)\, |l,m\rangle} = l(l+1) - m(m\pm 1)$$

We choose the phase convention $\quad c_\pm = \sqrt{l(l+1) - m(m\pm 1)}$

$$\therefore \boxed{\hat{l}_\pm |l,m\rangle = \sqrt{l(l+1) - m(m\pm 1)}\; |l, m\pm 1\rangle}$$