Physics 492: Quantum Mechanics II
Lecture 12: Introduction to Spin

Up until now we have be dealing with "motional" degrees of freedom. Such degrees of freedom have analogs in both classical and quantum physics in terms of canonical coordinates $(\vec{x}, \vec{p})$. However, there are degrees that are purely quantum mechanical, with no classical analog. The most important of these is spin which has huge implications for fundamental particle physics as well a mathematical formalism that underpins the foundation of quantum information science. The importance of spin cannot be overstated, and we will spend the remainder of the semester discussing it.

Historically, spin was discovered in the context of atomic spectroscopy. Uhlenbeck & Goudsmit introduced a "spining electron" in order explain the anamolous Zeeman effect. Wolfgang Pauli used this to explain his phenomenological "exclusion principle" which introduced the need for an addition "quantum number" in order to explain the periodic table of elements.

Mathematically, we could have predicted the existence of a "nonclassical solution" to quantum angular momentum. Let us define "abstract" angular momentum operators $\hat{J}_i$ that satisfy the commutation relations

$$[\hat{J}_i, \hat{J}_j] = i\hbar\, \epsilon_{ijk}\, \hat{J}_k \qquad [\hat{\vec{J}}^2, \hat{J}_i] = 0$$

We will define dimensionless operators $\hat{j}_i = \frac{\hat{J}_i}{\hbar}$. $[\hat{j}_i, \hat{j}_j] = i\,\epsilon_{ijk}\hat{j}_k$. The angular momentum raising and lowering operators $\hat{j}_\pm = \hat{j}_x \pm i\hat{j}_y$ satisfy commutation relations

$$[\hat{j}_+, \hat{j}_-] = 2\hat{j}_z, \qquad [\hat{j}_z, \hat{j}_\pm] = \pm\hat{j}_\pm$$

The standendard basis of of angular momentum are simultaneous eigenstates of $\hat{j}^2$ and $\hat{j}_z$. It follows from the commutation relations that $\hat{j}^2|j,m\rangle = j(j+1)|j,m\rangle$, $\hat{j}_z|j,m\rangle = m|j,m\rangle$. It also follows from the commutation relations that $\hat{j}_z(\hat{j}_\pm|j,m\rangle) = (m\pm1)(\hat{j}_\pm|j,m\rangle)$. And since $|m| \le j$, $\exists$ m_{max} and m_{min} such that $m_{max} = -m_{min} = j$, so m goes from m_{max} to m_{min} in integer steps $\Rightarrow$ There are $2j+1$ $|j,m\rangle$ states for a given j. This algebra implies $2j+1$ is and integer. This implies two possibilities

(1) j is an integer

(2) j is a half-integer: $\frac{n}{2}$ (n odd)

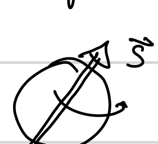
For j = integer m = integer. For j = half-integer, m = half-integer. However, we saw that the wave functions of $\hat{L}_z$ are $\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}$. And since $\Phi_m(\phi+2\pi) = \Phi_m(\phi)$ as the wave function must be single valued, for angular momentum associated with the motion of a particle (which we can orbital angular momentum), m must be an integer, and thus for orbital angular momentum, j must be an integer.

The algebraic solutions with $j = \frac{1}{2}$-integer and not associated with orbital motion. They represent an abstract angular momentum that we call spin angular momentum.

Thus the possible values for j are: $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \ldots$ The half-integer values can only be associated with spin angular momentum. The orbital-angular momentum l, can only be associated with integer j. The corresponding wave functions are the spherical harmonics $Y_{l,m}(\theta, \phi)$. There is nothing that forbids integer spin! In fact, there are two "species" of fundamental particles

- Fermions: Half-integer
- Bosons: Whole-integer

In the "standard model" matter is fermions (leptons and quarks). The forces are fields whose quanta are boson (e.g. photons are spin-1). We think of spin as an "intrinsic property" of the particle, like its mass or its charge. The difference here is that the spin is not a scalar. It is a "vector" quantity attached to the particle.



We typically denote $\hat{\vec{S}}$ as the "spin angular momentum;" we use $\hat{\vec{J}}$ as a generic angular momentum, spin or orbital (or both as we will see). Again, we should not think of the particle as a "spinning ball." There is no motion associated with spin.

Spin-$\frac{1}{2}$

The most fundamental unit of spin angular momentum is spin $s=\frac{1}{2}$. Fundamental particles likes the electron has spin $\frac{1}{2}$. The proton (composite of quarks and gluons) also has spin-$\frac{1}{2}$. Let us write the angular momentum matrices in the "standard basis"

$$\{|s,m_s\rangle\} = \{|\tfrac{1}{2},\tfrac{1}{2}\rangle, |\tfrac{1}{2},-\tfrac{1}{2}\rangle\} \; : \quad \text{Two-dimensional Hilbert space for } s=\tfrac{1}{2}$$

(ordered basis)

$$\Rightarrow \quad \hat{S}^2|s,m_s\rangle = \hbar^2\sqrt{s(s+1)}\,|s,m_s\rangle, \qquad \hat{S}_z = \hbar m_s |s,m_s\rangle$$

In this subspace, $\hat{S}_z \doteq \begin{bmatrix} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} \end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

To find $\hat{S}_x$ and $\hat{S}_y$, use $\hat{S}_x = \dfrac{\hat{S}_+ + \hat{S}_-}{2}$, $\hat{S}_y = \dfrac{\hat{S}_+ - \hat{S}_-}{2i}$ $\qquad \hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y$

$$\hat{S}_\pm |s,m_s\rangle = \hbar\sqrt{s(s+1) - m_s(m_s\pm 1)}\;|s,m_s\pm 1\rangle$$

$$\Rightarrow \hat{S}_+|\tfrac{1}{2},\tfrac{1}{2}\rangle = 0, \quad \hat{S}_+|\tfrac{1}{2},-\tfrac{1}{2}\rangle = \hbar\sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1)+\tfrac{1}{2}(\tfrac{1}{2})}\;|\tfrac{1}{2},\tfrac{1}{2}\rangle = \hbar|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$$\Rightarrow \quad \hat{S}_+ = \begin{bmatrix} 0 & \hbar \\ 0 & 0 \end{bmatrix} = \hbar\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \hat{S}_- = \hat{S}_+^{\dagger} = \hbar\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \hat{S}_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \hat{S}_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ +i & 0 \end{bmatrix}$$

$$\boxed{\begin{array}{l} \text{We define the } \underline{\text{Pauli Matrices}} \qquad \hat{S}_i = \frac{\hbar}{2}\hat{\sigma}_i \\[1em] \Rightarrow \hat{\sigma}_x \doteq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \hat{\sigma}_y \doteq \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \hat{\sigma}_z \doteq \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{array}}$$

The Pauli matrices have a critical role in modern quantum mechanics, particularly in the context of quantum information science, as we will see later in the semester.

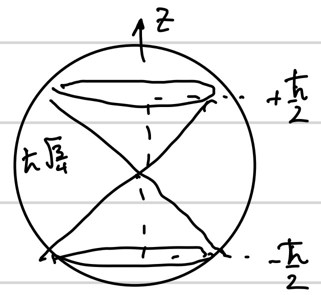
For simplicity, we call the "standard basis" vectors for spin-$\frac{1}{2}$

Spin-up: $|\uparrow\rangle \equiv |s=\tfrac{1}{2}, m_s=\tfrac{1}{2}\rangle$ $\qquad$ Spin-down $|\downarrow\rangle = |s=\tfrac{1}{2}, m=-\tfrac{1}{2}\rangle$

We define $\hat{\sigma}_+|\downarrow\rangle = |\uparrow\rangle$, $\hat{\sigma}_-|\uparrow\rangle = |\downarrow\rangle$, $\hat{\sigma}_+ \doteq \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\hat{\sigma}_- = \hat{\sigma}_+^{\dagger} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

Note, $\hat{\sigma}_{\pm} = \frac{\hat{S}_{\pm}}{\hbar}$ (without the factor of 2).

Note, of course, by the uncertainty principle, spin-up along z has uncertainty in the spin-projection along x and y. "spin-up" along z is only an eigenstate of $\hat{S}_z$



"Space Quantization" for spin-1/2

There are a number of important algebraic properties satisfied by the Pauli matrices. Most important amongst this is the following:

$$\boxed{\hat{\sigma}_i \hat{\sigma}_j = i \epsilon_{ijk} \hat{\sigma}_k + \delta_{ij} \hat{\mathbb{1}}}$$

Thus $\hat{\sigma}_x^2 = \hat{\sigma}_y^2 = \hat{\sigma}_z^2 = \hat{\mathbb{1}}$ (all the Pauli matrices square to the identity)

$\hat{\sigma}_x \hat{\sigma}_y = i \hat{\sigma}_z$ and cyclic permutations. These are very useful relations.

$\hat{\sigma}_x \hat{\sigma}_y = -\hat{\sigma}_y \hat{\sigma}_x \Rightarrow \hat{\sigma}_x \hat{\sigma}_y + \hat{\sigma}_y \hat{\sigma}_x = 0$ "anti-commute"

As we have seen in the Problem Sets, we can diagonalize and find the eigenvalues and eigenvectors of $\hat{\sigma}_x$ and $\hat{\sigma}_y$

$\hat{\sigma}_x$: Eigenvalues ± 1, Eigenvectors $|\uparrow_x\rangle = \frac{|\uparrow_z\rangle + |\downarrow_z\rangle}{\sqrt{2}}$, $|\downarrow_x\rangle = \frac{|\uparrow_z\rangle - |\downarrow_z\rangle}{\sqrt{2}}$

$\hat{\sigma}_y$: Eigenvalues ± 1, Eigenvectors $|\uparrow_y\rangle = \frac{|\uparrow_z\rangle + i|\downarrow_z\rangle}{\sqrt{2}}$, $|\downarrow_y\rangle = \frac{|\uparrow_z\rangle - i|\downarrow_z\rangle}{\sqrt{2}}$

Here I have written the standard basis as $\{|\uparrow_z\rangle, |\downarrow_z\rangle\}$ to emphasize that these are eigenvectors of $\hat{\sigma}_z$: $\hat{\sigma}_z|\uparrow_z\rangle = +|\uparrow_z\rangle$, $\hat{\sigma}_z|\downarrow_z\rangle = -|\downarrow_z\rangle$

Of course the eigenvectors of a given Hermitian operator associated with different eigenvalues are orthogonal.

$\langle\uparrow_z|\downarrow_z\rangle = 0$, $\langle\uparrow_x|\downarrow_x\rangle = 0$, $\langle\uparrow_y|\downarrow_y\rangle = 0$

However, the eigenstates of $\hat{\sigma}_z$ and those of $\hat{\sigma}_x$ and of $\hat{\sigma}_y$ are not orthogonal.

$$\langle \uparrow_z | \uparrow_x \rangle = \langle \downarrow_z | \uparrow_x \rangle = \frac{1}{\sqrt{2}} \qquad \langle \uparrow_z | \downarrow_x \rangle = -\langle \downarrow_z | \downarrow_x \rangle = \frac{1}{\sqrt{2}}$$

$$\langle \uparrow_z | \uparrow_y \rangle = \langle \uparrow_z | \downarrow_y \rangle = \frac{1}{\sqrt{2}} \qquad \langle \downarrow_z | \uparrow_y \rangle = -\langle \downarrow_z | \downarrow_y \rangle = \frac{i}{\sqrt{2}}$$

Thus, for example, if one prepared the spin as spin-up along x $|\psi\rangle = |\uparrow_x\rangle$ and then did a projective measurement of $\hat{\sigma}_z$, one would find $|\uparrow_z\rangle$ and $|\downarrow_z\rangle$ with probability $P_{\uparrow_z} = |\langle \uparrow_z | \psi \rangle|^2 = \frac{1}{2}$, $P_{\downarrow_z} = |\langle \downarrow_z | \psi \rangle|^2 = \frac{1}{2}$
The same is true for the system prepared along $|\downarrow_x\rangle$, $|\uparrow_y\rangle$, or $|\downarrow_y\rangle$.
In all cases, from the probability amplitudes above, measuring $\hat{\sigma}_z$ with yield $|\uparrow_z\rangle$ or $|\downarrow_z\rangle$ is 50% chance. Similarly, I leave is as an exercise to show that if you prepare $|\psi\rangle = |\uparrow_x\rangle$ and measure $\hat{\sigma}_y$ you will find $|\uparrow_y\rangle$ or $|\downarrow_y\rangle$ with 50-50 probability.