Physics 492: Quantum Mechanics II

Lecture 13: Larmor Precession of a Spin in a Magnetic field

Spin and Magnetic Interactions

If the electron were a spinnin ball of charge, it would have an intrinsic magnetic dipole moment $\hat{\mu}_{spin} \overset{?}{=} -\mu_B \left(\frac{\hat{S}}{\hbar}\right)$. However the spin is not due to the orbital rotation of the electron. The existence of its magnetic dipole was revealed in the anamolous Zeeman effect, but in order to fit the data Landé introduced a fudge factor now known as the Landé g-factor.

The actual "intrinsic" magnetic dipole moment associated with spin of the negatively charge particle is:

$$\hat{\mu}_{spin} = -g_s \mu_B \frac{\hat{S}}{\hbar} \quad ; \quad \text{for spin-}\tfrac{1}{2} \quad \underline{g_s = 2}$$

The origin of the g-factor of the electron is explained in Dirac's relativistic theory: The Dirac equation. Verifying g_s is one of the great triumphs of Quantum Electrodynamics.

Consider the interaction of an electron with a magnetic field, when the orbital angular momentum is $l=0$ (we'll consider later the case of both orbital & spin). The interaction Hamiltonian of a spin in a magnetic field is

$$\hat{H}_{int} = -\vec{\mu}_{spin} \cdot \vec{B} = g_s \mu_B \frac{\vec{S}}{\hbar} \cdot \vec{B} = g_s \mu_B \frac{\hat{\vec{\sigma}}}{2} \cdot \vec{B} = \mu_B \hat{\vec{\sigma}} \cdot \vec{B}$$

Having used $\hat{S} = \frac{\hbar}{2} \hat{\vec{\sigma}}$

Now let us choose the direction of $\vec{B}$ as the z-axis: $\vec{B} = B_0 \vec{e}_z$.

$$\Rightarrow \quad \hat{H}_{int} = \mu_B B_0 \hat{\sigma}_z$$

The energy eigenvectors are thus $|\uparrow_z\rangle$, $|\downarrow_z\rangle$ with eigenvalues $\pm\mu_B B_0$ respectively

Low energy state $|\downarrow_z\rangle$

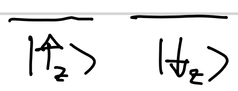
$\vec{B}_0 \uparrow \quad \langle\vec{\mu}_{spin}\rangle \uparrow$

High energy state $|\uparrow_z\rangle$

$\vec{B}_0 \uparrow \quad \langle\hat{\vec{\mu}}_{spin}\rangle \downarrow$

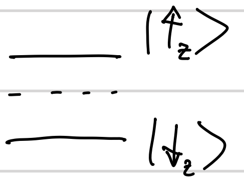
Energy-level diagram:

$B_0 = 0$



Degenerate

$B_0 > 0$

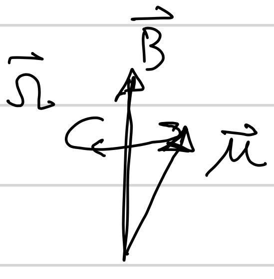


"Zeeman splitting"

The energy splitting is associated with a "Bohr frequency" $\Omega = \frac{E_\uparrow - E_\downarrow}{\hbar} = \frac{2\mu_B B_0}{\hbar}$

Larmor Precession

Consider again a spin in a magnetic field along z.
If the system does not start in a stationary state, i.e., $|\uparrow_z\rangle$ or $|\downarrow_z\rangle$, it will evolve as a function of time. Classically, the magnetic moment will experience a torque, and the gyroscopic action will cause the angular momentum to precess around the axis:



Torque: $\vec{\tau} = \vec{\mu} \times \vec{B}$

If $\vec{\mu} = \gamma\vec{J}$

$\vec{J}$ = angular momentum
γ = gyromagnetic ratio

$\Rightarrow \vec{\tau} = \frac{d\vec{J}}{dt} = \vec{J} \times \vec{\Omega} \quad (\vec{\Omega} = \gamma\vec{B})$

The angular momentum precesses around $\vec{B}$ with freq $\Omega = \gamma B$

The same is true quantum mechanically. As we will see, with Ehrenfest's theorem, for an electron with $\hat{H}_{int} = -g_s\mu_B \frac{\hat{\vec{S}}}{\hbar} \cdot \vec{B}$

$$\frac{d}{dt}\langle\hat{\vec{S}}\rangle = \vec{\Omega} \times \langle\hat{\vec{S}}\rangle, \quad \text{where } \vec{\Omega} = 2\mu_B\vec{B}/\hbar \quad \left(\begin{array}{c}\text{Here } \gamma < 0 \\ \text{since charge} < 0\end{array}\right)$$

$$|\vec{\Omega}| = 2\mu_B B_0/\hbar$$

The mean spin of the electron will precess around the magnetic field at the Larmor frequency $\Omega = 2\mu_B B/\hbar$.

We can solve this also by seeing how the state evolves as a function of time. Consider a spin initially prepared at $t=0$ as spin-up along x.

$$|\psi(0)\rangle = |\uparrow_x\rangle$$

Suppose a magnetic field $\vec{B} = B_0 \vec{e}_z$ is switched on along z. As we have seen, the Hamiltonian is $\hat{H}_{int} = \mu_B B_0 \hat{\sigma}_z = \frac{\hbar\Omega}{2}\hat{\sigma}_z$. We seek $|\psi(t)\rangle$. One way we have seen to solve the initial value problem is to first expand $|\psi(0)\rangle$ in terms of the eigenstates of the Hamiltonian, $|\uparrow_z\rangle$ and $|\downarrow_z\rangle$

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}|\uparrow_z\rangle + \frac{1}{\sqrt{2}}|\downarrow_z\rangle$$

Then at a later time $|\psi(0)\rangle = \frac{1}{\sqrt{2}} e^{\frac{-iE_\uparrow t}{\hbar}} |\uparrow_z\rangle + \frac{1}{\sqrt{2}} e^{-i\frac{E_\downarrow t}{\hbar}} |\downarrow_z\rangle$

$$= \frac{1}{\sqrt{2}} e^{\frac{-i\Omega t}{2}} |\uparrow_z\rangle + \frac{1}{\sqrt{2}} e^{\frac{-i\Omega t}{2}} |\downarrow_z\rangle$$

$$\doteq \frac{1}{\sqrt{2}} \begin{bmatrix} e^{\frac{-i\Omega t}{2}} \\ e^{\frac{i\Omega t}{2}} \end{bmatrix} \quad \text{(in the standard basis)}$$

Let us calculate $\langle \hat{\vec{S}} \rangle(t) = \frac{\hbar}{2}\langle\psi(t)|\hat{\vec{\sigma}}|\psi(t)\rangle$

$$\langle\psi(t)|\hat{\sigma}_x|\psi(t)\rangle = \frac{1}{2}\begin{bmatrix} e^{\frac{i\Omega t}{2}} & e^{\frac{-i\Omega t}{2}} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e^{\frac{-i\Omega t}{2}} \\ e^{\frac{i\Omega t}{2}} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{\frac{i\Omega t}{2}} & e^{\frac{-i\Omega t}{2}} \end{bmatrix} \begin{bmatrix} e^{\frac{i\Omega t}{2}} \\ e^{\frac{-i\Omega t}{2}} \end{bmatrix}$$

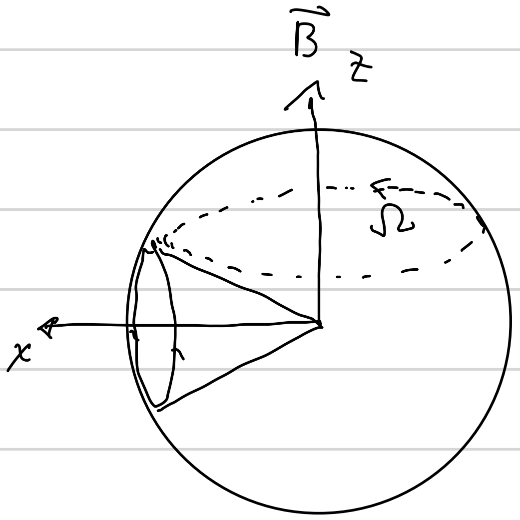
$$= \frac{1}{2}\left(e^{i\Omega t} + e^{-i\Omega t}\right) = \cos\Omega t$$

$$\langle\psi(t)|\hat{\sigma}_y|\psi(t)\rangle = \frac{1}{2}\begin{bmatrix} e^{\frac{i\Omega t}{2}} & e^{\frac{-i\Omega t}{2}} \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} e^{\frac{-i\Omega t}{2}} \\ e^{\frac{i\Omega t}{2}} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{\frac{i\Omega t}{2}} & e^{\frac{-i\Omega t}{2}} \end{bmatrix} \begin{bmatrix} -i e^{\frac{i\Omega t}{2}} \\ i e^{\frac{-i\Omega t}{2}} \end{bmatrix}$$

$$= \frac{1}{2}\left(-i e^{i\Omega t} + i e^{-i\Omega t}\right) = \sin\Omega t$$

$$\langle\psi(t)|\hat{\sigma}_z|\psi(t)\rangle = \frac{1}{2}\begin{bmatrix} e^{\frac{i\Omega t}{2}} & e^{\frac{-i\Omega t}{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} e^{\frac{-i\Omega t}{2}} \\ e^{\frac{i\Omega t}{2}} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{\frac{i\Omega t}{2}} & e^{\frac{-i\Omega t}{2}} \end{bmatrix} \begin{bmatrix} e^{\frac{i\Omega t}{2}} \\ -e^{\frac{-i\Omega t}{2}} \end{bmatrix} = 0$$

$$\Rightarrow \quad \langle \hat{\vec{S}} \rangle(t) = \frac{\hbar}{2}\left(\cos\omega t\, \vec{e}_x + \sin\omega t\, \vec{e}_y\right)$$



The spin precesses around the $\vec{B}$ field at the Larmor frequency

We see this precession directly in the state itself:

$$|\psi(t)\rangle \doteq \frac{1}{\sqrt{2}}\begin{bmatrix} e^{-i\frac{\Omega t}{2}} \\ e^{\frac{i\Omega t}{2}} \end{bmatrix}$$

- At $\Omega t = \frac{\pi}{2}$ $\quad |\psi_{\frac{\pi}{2}}\rangle \doteq \frac{1}{\sqrt{2}}\begin{bmatrix} e^{-\frac{i\pi}{4}} \\ e^{\frac{i\pi}{4}} \end{bmatrix} = \frac{e^{-\frac{i\pi}{4}}}{\sqrt{2}}\begin{bmatrix} 1 \\ e^{\frac{i\pi}{2}} \end{bmatrix} = e^{-\frac{i\pi}{4}}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} \doteq \overset{\text{overall phase irrelevant}}{\boxed{e^{-\frac{i\pi}{4}}}}|\uparrow_y\rangle \equiv |\uparrow_y\rangle$

Thus after a $\frac{1}{4}$ period (rotation angle $\frac{2\pi}{4} = \frac{\pi}{2}$), the state of the spin is $|\psi\rangle = |\uparrow_y\rangle$ (spin-up along y)

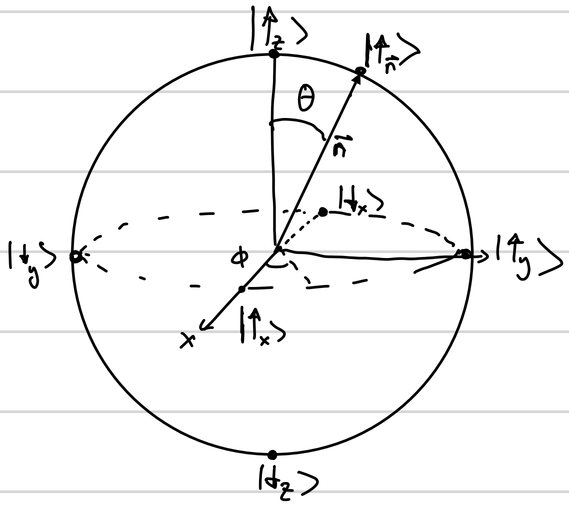
- At $\Omega t = \frac{\pi}{2}$ $\quad |\psi_{\pi}\rangle \doteq \frac{1}{\sqrt{2}}\begin{bmatrix} e^{-\frac{i\pi}{2}} \\ e^{\frac{i\pi}{2}} \end{bmatrix} = \frac{e^{-\frac{i\pi}{2}}}{\sqrt{2}}\begin{bmatrix} 1 \\ e^{i\pi} \end{bmatrix} = e^{-\frac{i\pi}{4}}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} \doteq \overset{\text{overall phase irrelevant}}{\boxed{e^{-\frac{i\pi}{2}}}}|\downarrow_x\rangle \equiv |\downarrow_x\rangle$

Thus after a $\frac{1}{2}$ period (rotation angle $\frac{2\pi}{2} = \pi$), the state of the spin is $|\psi\rangle = |\uparrow_x\rangle$ (spin-up along x)

Thus, spin state precesses. $|\psi(0)\rangle = e^{i\gamma(t)}|\uparrow_{\vec{n}(t)}\rangle$: Spin up along $\vec{n}(t)$ where $\vec{n}(t) = \vec{e}_x \cos\Omega t + \vec{e}_y \sin\Omega t$.

As we will see in homework, every state of a spin $-1/2$ $|\psi\rangle \equiv |\uparrow_{\vec{n}}\rangle$ for some some choice of $\vec{n}$

$\Rightarrow$ $|\psi\rangle$ is in one-to-one correspondence with a direction $\vec{n} = (\theta, \phi)$ on sphere. The mapping is known as the Bloch sphere, after Felix Bloch for his work on magnetic manipulations of spins.



Bloch Sphere

Directions on a sphere
$\Rightarrow$ Hibert spin of spin-$1/2$

Note: Antipodal points on the Bloch sphere correspond to orthogonal states in Hilbert space. The north pole is $|\uparrow_z\rangle$, the south is $|\downarrow_z\rangle$, $\langle\uparrow_z|\downarrow_z\rangle = 0$.
All of the states in the equator are 50-50 superpositions of $|\uparrow_z\rangle$ & $|\downarrow_z\rangle$.
In general, for a state in the equation $|\uparrow\rangle_\phi = \frac{1}{\sqrt{2}}(|\uparrow_z\rangle + e^{i\phi}|\downarrow_z\rangle)$,
where ϕ is the azimuthal angle.
More generally, as you will show in homework $|\uparrow_{\vec{n}}\rangle = \cos\frac{\theta}{2}|\uparrow_z\rangle + e^{i\phi}\sin\frac{\theta}{2}|\downarrow_z\rangle$.
Moreover, $|\langle\uparrow_{\vec{n}'}|\uparrow_{\vec{n}}\rangle|^2 = \frac{1+\vec{n}\cdot\vec{n}'}{2} = \frac{1+\cos\Theta}{2} = \cos^2\frac{\Theta}{2}$, where $\vec{n}\cdot\vec{n}' = \cos\Theta$