Lecture 13: The Stern-Gerlach Experiment – Quantum Measurement Revisited

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From early in our exploration of quantum mechanics, measurement played a central role. A basic "axiom" of quantum theory is the "von Neumann projection postulate":

> If an observable $\hat{A}$ is measured, the result is one of its eigenvalues, a. After the measurement, the system is "projected" into the eigenvector $|a\rangle$. If $|\psi\rangle$ is the state before the measurement, the probability of this occurance is $|\langle a|\psi\rangle|^2$.

But, we have not actually explored any physical examples of these measurements. The most fundamental example is the measurement of the angular momentum component of a spin-1/2 particle which takes on only two possible values. This measurement was first carried out by Stern and Gerlach in 1922 to test Bohr's "space quantization". This was before Uhlenbeck and Goudsmidt's ~~[illegible]~~ invention of spin angular momentum, so Stern and Gerlach's results ~~[illegible]~~ were not completely understood. Nonetheless, the results were startling and one of the first real pictures of the strange quantum world.

Stern-Gerlach experiment

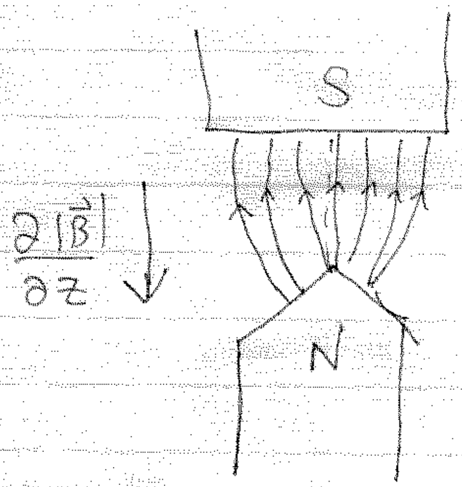
Consider the force on a magnetic moment which moves through a spatially <u>inhomogeneous</u> magnetic field.

The potential energy $V = -\vec{\mu}\cdot\vec{B}(\vec{r})$

Thus the force $\vec{F} = -\vec{\nabla}V = \vec{\nabla}(\vec{\mu}\cdot\vec{B}(\vec{r}))$

$$= (\vec{\mu}\cdot\vec{\nabla})\,\vec{B}(\vec{r})$$

Stern and Gerlach set up a spatially inhomogeneous field with one very large component (call it the z-component)



$\vec{B} \approx B(z)\,\vec{e}_z$ + (small x, y comownpent)

$$\Rightarrow (\vec{\mu}\cdot\vec{\nabla})\,\vec{B} \approx \mu_z \frac{\partial B}{\partial z}\,\vec{e}_z$$

Now quantum mechanically, $\hat{\vec{\mu}} = \gamma_J \hat{\vec{J}}$

$$\Rightarrow \boxed{\hat{\vec{F}} = \left(\gamma_J \frac{\partial B}{\partial z}\right) \hat{J}_z\, \vec{e}_z}$$

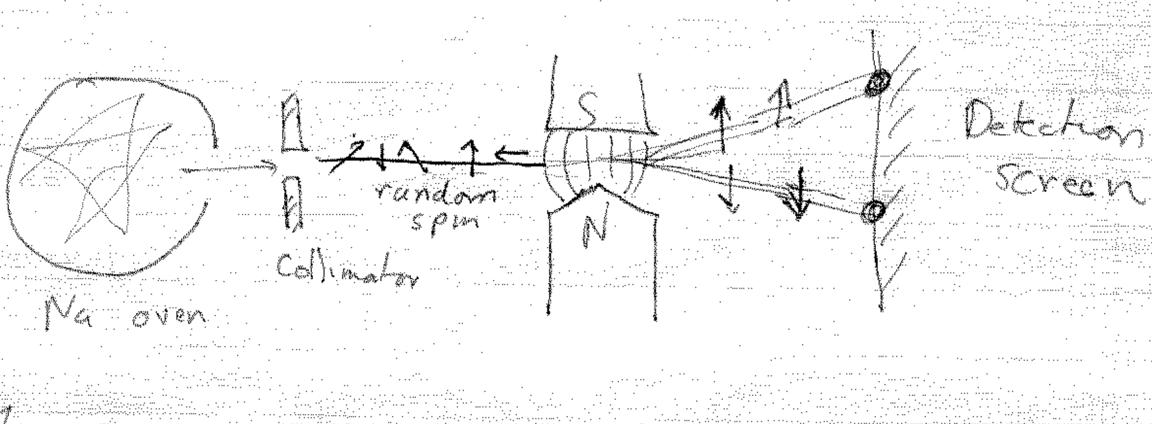
Stern and Gerlach sent Na (sodium) atom from an oven, columated into a beam, into the space between the magnet poles.

Na is an alkali atom with one valence electron with orbital angular momentum $l = 0$ ("s-state"). Thus all of the angular momentum of the atom is due to the spin angular momentum of the single valence electron spin-$\frac{1}{2}$

$$\Rightarrow \hat{\vec{\mu}} = -2 \mu_B \hat{\vec{S}}/\hbar = -\mu_B \hat{\vec{\sigma}}$$

$$\Rightarrow \text{Force} \quad \boxed{\hat{\vec{F}} = -\mu_B \frac{\partial B_z}{\partial z} \hat{\sigma}_z}$$

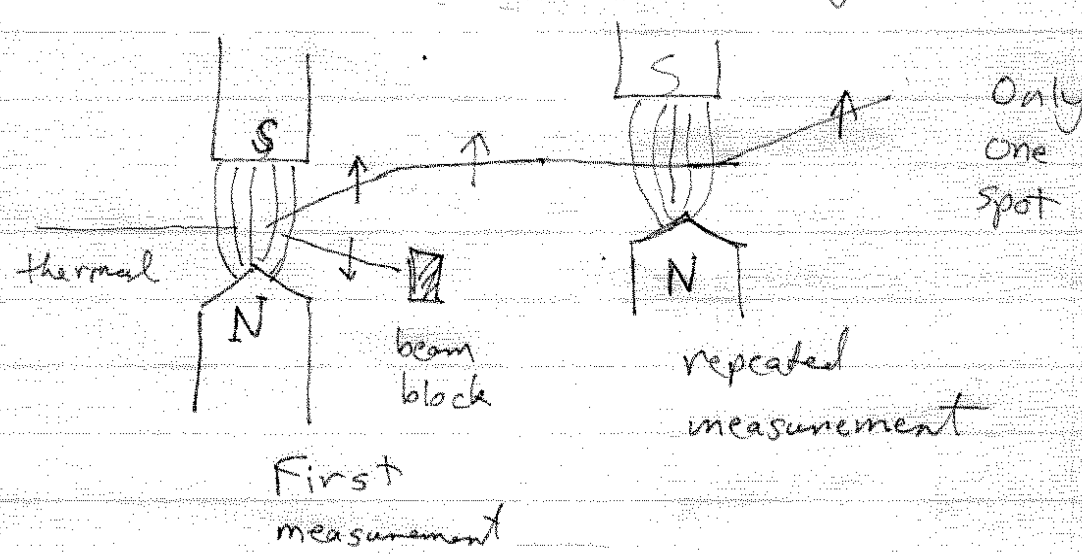
The force thus takes two eigenvalues; the "spin-up atoms" $|\uparrow_z\rangle$ will experience a different force from "spin-down" $|\downarrow_z\rangle$, thereby splitting the beam in two.



The spins which emerge from the oven have a random orientation

The S-G experiment has all the essential ingredients of a quantum measurement. The quantum "degree of freedom", here spin-1/2, is correlated with a macro-scopic degree of freedom, here the position of the atom. Once the two spots are "resolvable", we can measure the spin state by determining which beam the atom is in.

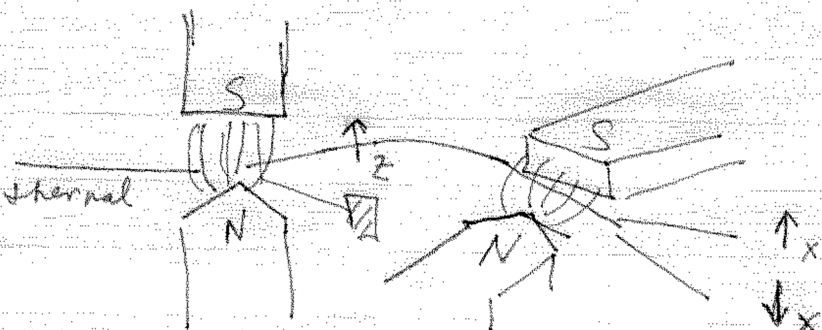
State preparation: We can prepare a state by measuring an observable and finding one of the eigenvalues. Afterwards, by the von Neumann projection postulate, the state is the corresponding eigenvector.



By redirecting the "spin-up" beam into a second S-G apparatus we are guaranteed to find the eigenstate we prepared. Thus there is only one spot, whereas with the thermal input there were two.

Sequential Measurements of Noncompatible Observables

Suppose the second apparatus is oriented to have its gradient along the x-axis



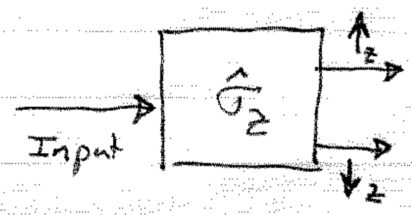
The state $|\uparrow_z\rangle$ is not an eigenstate of $\hat{\sigma}_x$.

We have found
$$|\uparrow_x\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle + |\downarrow_z\rangle\right)$$
$$|\downarrow_x\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle - |\downarrow_z\rangle\right)$$

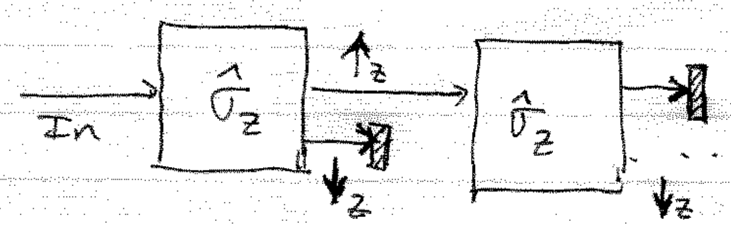
or inverting
$$|\uparrow_z\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_x\rangle + |\downarrow_x\rangle\right)$$
$$|\downarrow_z\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_x\rangle - |\downarrow_x\rangle\right)$$

Thus $|\uparrow_z\rangle$ is a 50-50 superposition of $|\uparrow_x\rangle$ and $|\downarrow_x\rangle$. Even though it is a "pure quantum state", not all measurements yield definite values. 50% of the time the spin-up atom along z with be found with spin-up along x and 50% spin-down along x.

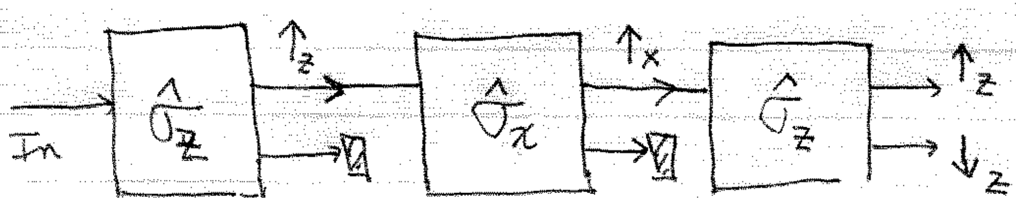
Simplified picture. Let me denote the S-G apparatus with gradient field along, e.g., the z-axis as a "black box" with two output ports



Consider then the following sequence



Because we perform repeated measurements on the same observable, there is zero probability of finding $|\downarrow_z\rangle$ after the second measurement; we completely "block the beam". This is analogous to cross-polarizers in optics. Suppose, however, we place a $\hat{\sigma}_x$ S-G apparatus in between



This is analogous to placing a polarizer at 45° between the cross polarized.

Since $|\uparrow_x\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle + |\downarrow_z\rangle\right)$, there is a $\frac{1}{2}$ probability of detecting $|\downarrow_z\rangle$ whereas with the $\boxed{\hat{\sigma}_x}$ measurement the probability was <u>zero</u>.

This is, of course, the difference between quantum <u>probability amplitudes</u> and classical <u>probabilities</u>.

In the cascaded measurement, $\boxed{\sigma_z} \xrightarrow{\uparrow_z} \boxed{\sigma_x} \xrightarrow{\uparrow_x} \boxed{\sigma_z} \rightarrow \downarrow_z$,

The probability of finding $|\downarrow_z\rangle$ is the product of conditional probabilities for uncorrelated events

$$p(\downarrow_z|\uparrow_x,\uparrow_z) = \underbrace{p(\downarrow_z|\uparrow_x)}_{|\langle\downarrow_z|\uparrow_x\rangle|^2 = \frac{1}{2}} \; \underbrace{p(\uparrow_x|\uparrow_z)}_{|\langle\downarrow_x|\uparrow_z\rangle|^2 = 1/2}$$

Now <u>without</u> the intermediate apparatus, <u>classical theory</u> would argue as follows. If we don't measure $\hat{\sigma}_x$ we must sum the <u>probabilities</u> of all possible of all alternatives

$$\Rightarrow p(\downarrow_z|\uparrow_z) = \sum_{m_x=\pm 1} \underbrace{p(\downarrow_z|m_x)}_{=1/2} \; \underbrace{p(m_x|\uparrow_z)}_{=1/2}$$

$$= 2\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = 1/2$$

<u>Not correct quantum result</u>

Quantum mechanically, if the different possibilities are indistinguishable (i.e. there is no information available to distinguish the different alternatives) we must add probability amplitudes. These different alternatives can interfere.

$$p(\downarrow_z | \uparrow_z) = |\langle \downarrow_z | \uparrow_z \rangle|^2 =$$

insert complete set $|\uparrow_x\rangle\langle\uparrow_x| + |\downarrow_x\rangle\langle\downarrow_x| = \mathbb{1}$

$$\Rightarrow p(\downarrow_z|\uparrow_z) = |\langle\downarrow_z|\uparrow_x\rangle\langle\uparrow_x|\uparrow_z\rangle + \langle\downarrow_z|\downarrow_x\rangle\langle\downarrow_x|\uparrow_z\rangle|^2$$

$$= |\langle\downarrow_z|\uparrow_x\rangle|^2 |\langle\uparrow_x|\uparrow_z\rangle|^2 + |\langle\downarrow_z|\downarrow_x\rangle|^2 |\langle\downarrow_x|\uparrow_z\rangle|^2$$

$$+ \left(\langle\downarrow_z|\uparrow_x\rangle\langle\uparrow_x|\uparrow_z\rangle \langle\downarrow_z|\downarrow_x\rangle^* \langle\downarrow_x|\uparrow_z\rangle^* + \text{c.c.}\right)$$

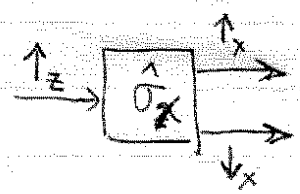
$$= \underbrace{p(\downarrow_z|\uparrow_x)\, p(\uparrow_x|\downarrow_z) + p(\downarrow_z|\downarrow_x)\, p(\downarrow_x|\uparrow_z)}_{\text{Classical terms}}$$

+ Interference terms

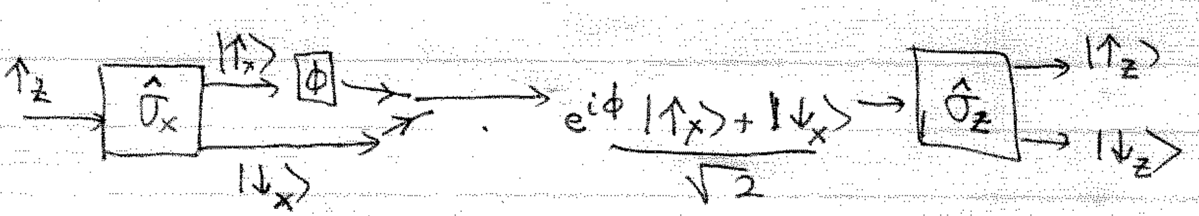
$= 0$ ✓

What constitutes a measurement?

Suppose we send a $|\uparrow_z\rangle$ state into a $\boxed{\hat{\sigma}_x}$ S-G apparatus



There is a 50-50 probability of the atom emerging in the $|\uparrow_x\rangle$ or $|\downarrow_x\rangle$ port. But if don't detect which port the spin exited from, did we measure $\hat{\sigma}_x$? In a sense, no. In principle we can recombine these ~~between~~ beams, with a phase shift in one arm.



Probabilities: $P_{\uparrow_z} = \left|\dfrac{e^{i\phi}+1}{2}\right|^2 = \cos^2 \phi/2$

$$P_{\downarrow_z} = \left|\frac{e^{i\phi}-1}{2}\right|^2 = \sin^2 \phi/2$$

So a measurement is something that removes coherence, i.e. the ability for quantum processes to interfere.

More on this later.