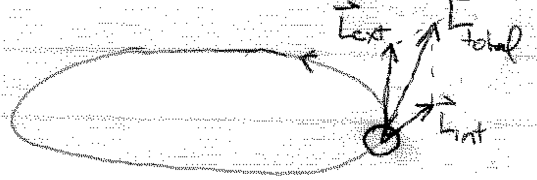
Physics 492 - Quantum II

Lecture 16: Addition of Angular Momentum

We have that angular momentum plays a central role in quantum mechanics, not the least of which is the intrinsics spin. So far we have treated spin and orbital angular momentum separately, but what about the total angular momentum?

Classically, the angular momentum adds as a vector



E.g. Spinning earth orbiting the sun is the sum of "internal" $\vec{L}_{int}$ and "external" $\vec{L}_{ext}$

$$\vec{L}_{total} = \vec{L}_{int} + \vec{L}_{ext}$$

Quantum Mechanically, the situation is more complex since the different components of angular momentum don't commute

We must therefore be careful we defining states with composite angular momentum

Example: Two spin ½ particles

$$\mathcal{H}_{AB} = h \otimes h = h^{\otimes 2}$$

h spanned by $\{|\uparrow\rangle, |\downarrow\rangle\}$ ← Standard basis

Basis defined ~~..~~ as the simultaneous eigenstates of $\hat{\vec{S}}^2$ and $\hat{S}_z$

The total (composite) Hilbert space is spanned by the product states

$$\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$$

These states are the simultaneous eigenvectors of

$$\{\hat{\vec{S}}_A^2, \hat{S}_{z_A}, \hat{\vec{S}}_B^2 \ \hat{S}_{z_B}\}$$

Consider now the total angular momentum operator

$$\hat{\vec{S}} = \hat{\vec{S}}_A + \hat{\vec{S}}_B = \hat{\vec{S}}_A \otimes \hat{\mathbb{1}}_B + \hat{\mathbb{1}}_A \otimes \hat{\vec{S}}_B$$

$$\hat{S}_z = \hat{S}_{z_A} + \hat{S}_{z_B} = \hat{S}_{z_A} \otimes \hat{\mathbb{1}}_B + \hat{\mathbb{1}}_A + \hat{S}_{z_B}$$

$$\hat{\vec{S}}^2 = \hat{\vec{S}} \cdot \hat{\vec{S}} = \hat{\vec{S}}_A^2 + \hat{\vec{S}}_B^2 + 2\hat{\vec{S}}_A \cdot \hat{\vec{S}}_B$$

$$= \hat{\vec{S}}_A^2 \otimes \hat{\mathbb{1}}_B + \hat{\mathbb{1}}_A \otimes \hat{\vec{S}}_B^2$$

$$+2\left(\hat{S}_{x_A} \otimes \hat{S}_{x_B} + \hat{S}_{y_A} \otimes \hat{S}_{y_B} + \hat{S}_{z_A} \otimes \hat{S}_{z_B}\right)$$

Does the total angular momentum satisfy the expected commutation relation?

$$[\hat{S}_x, \hat{S}_y] = i\hat{S}_z \,?\;, \quad [\hat{\vec{S}}^2, \hat{S}_z] = 0\,?$$

<u>Check</u>: $\hat{S}_x = \hat{S}_{x_A} + \hat{S}_{x_B}$ etc.

- $[\hat{S}_x, \hat{S}_y] = [\hat{S}_{x_A} + \hat{S}_{x_B}, \hat{S}_{y_A} + \hat{S}_{y_B}]$

$$= [\hat{S}_{x_A}, \hat{S}_{y_A}] + [\hat{S}_{x_B}, \hat{S}_{y_B}] + \underbrace{[\hat{S}_{x_A}, \hat{S}_{y_B}] + [\hat{S}_{x_B}, \hat{S}_{y_A}]}_{=0}$$

(operators on different d.o.f. commute)

$$\Rightarrow [\hat{S}_x, \hat{S}_y] = i\hat{S}_{z_A} + i\hat{S}_{z_B} = i\hat{S}_z \checkmark$$

and cyclic permutations

- $[\hat{\vec{S}}^2, \hat{S}_z] = [\hat{\vec{S}}_A^2 + \hat{\vec{S}}_B^2 + 2\hat{\vec{S}}_A\cdot\hat{\vec{S}}_B, \hat{S}_{z_A} + \hat{S}_{z_B}]$

$$= \cancel{[\hat{\vec{S}}_A^2, \hat{S}_{z_A}]}^{0} + \cancel{[\hat{\vec{S}}_B^2, \hat{S}_{z_B}]}^{0} + $$

$$+2\left([\hat{\vec{S}}_A\cdot\hat{\vec{S}}_B, \hat{S}_{z_A}] + [\hat{\vec{S}}_A\cdot\hat{\vec{S}}_B, \hat{S}_{z_B}]\right)$$

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<u>Aside:</u> $[\hat{\vec{S}}_A \cdot \hat{\vec{S}}_B, \hat{S}_{z_A}] = [\hat{S}_{x_A} \otimes \hat{S}_{x_B} + \hat{S}_{y_A} \otimes \hat{S}_{y_B} + \hat{S}_{z_A} \otimes \hat{S}_{z_A}),$
$\hat{S}_{zA} \otimes \hat{\mathbb{1}}_B]$

$$= \underbrace{[\hat{S}_{x_A}, \hat{S}_{z_A}]}_{-i\hat{S}_{yA}} \otimes \hat{S}_{x_B} + \underbrace{[\hat{S}_{y_A}, \hat{S}_{z_A}]}_{i\hat{S}_{x_A}} \otimes \hat{S}_{y_B}$$

$$[\hat{\vec{S}}_A \cdot \hat{\vec{S}}_B, \hat{S}_{z_B}] = \hat{S}_{x_A} \otimes \underbrace{[\hat{S}_{x_B}, \hat{S}_{z_B}]}_{-i\hat{S}_{y_B}} + \hat{S}_{y_A} \otimes \underbrace{[\hat{S}_{y_B}, \hat{S}_{z_B}]}_{i\hat{S}_{x_B}}$$

$$\Rightarrow [\hat{\vec{S}}_A \cdot \hat{\vec{S}}_B, \hat{S}_{z_A} + \hat{S}_{z_B}] = 0$$

$$\therefore [\hat{\vec{S}}^2, \hat{S}_z] = 0$$

Thus $\hat{\vec{S}} = \hat{\vec{S}}_A + \hat{\vec{S}}_B$ satisfies the usual commutations rules for angular momentum

$\Rightarrow \exists$ common eigenstates of $\hat{\vec{S}}^2$ and $\hat{S}_z$, $|S, M_S\rangle$

$$\hat{\vec{S}}^2 |S, M_S\rangle = S(S+1) |S, M_S\rangle$$
$$S_z |S, M_S\rangle = M_S |S, M_S\rangle$$

What is the relationship to the product bases?

<u>Note</u> :

The states $\{ |\uparrow\uparrow\rangle, \ |\uparrow\downarrow\rangle, \ |\downarrow\uparrow\rangle, \ |\downarrow\downarrow\rangle \}$ are simultaneous eigenvectors of

$$\{ \hat{\vec{S}}_A^2 , \ \hat{\vec{S}}_B^2 , \ \hat{S}_{z_A} , \ \hat{S}_{z_B} \}$$

The states $|S, M_S\rangle$ are simultaneous eigenstates of $\hat{\vec{S}}^2$ and $\hat{S}_z$

Now, $[\hat{\vec{S}}_A^2 , \hat{\vec{S}}] = \underbrace{[\hat{\vec{S}}_A^2 , \hat{\vec{S}}_A]}_{=0} + \underbrace{[\hat{\vec{S}}_B^2 , \hat{\vec{S}}_B]}_{=0}$

and similarly, $[\hat{\vec{S}}_B^2 , \vec{S}] = 0$

$\Rightarrow \quad [\hat{\vec{S}}^2 , \hat{\vec{S}}_A^2] = 0$ and $[\hat{\vec{S}}^2 , \hat{\vec{S}}_B^2]$

$\Rightarrow$ We can include s_A and s_B quantum numbers in our new basis

$$\{ |S, M_S ; s_A , s_B \rangle \}$$

Simultaneous eigenvectors of

$$\{ \hat{\vec{S}}^2 , \ \hat{S}_z , \ \hat{\vec{S}}_A^2 , \ \hat{\vec{S}}_B^2 \}$$

However $[\hat{\vec{S}}^2, \hat{S}_{z_A}] = 2[\hat{\vec{S}}_A \cdot \hat{\vec{S}}_B, \hat{S}_{z_A}] \neq 0$

$$[\hat{\vec{S}}^2, \hat{S}_{z_B}] \neq 0$$

$\Rightarrow$ Two different bases for $\mathcal{H}_{AB}$

- Product basis = "Uncoupled representation"
  Simultaneous eigenstates of $\{\hat{S}_A^2, \hat{S}_B^2, \hat{S}_{z_A}, \hat{S}_{z_B}\}$

  $$|s_A, m_{s_A}; s_B, m_{s_B}\rangle = |s_A, m_{s_A}\rangle \otimes |s_B, m_{s_B}\rangle$$

- "Coupled representation"

  Simultaneous eigenstates of $\{\hat{\vec{S}}^2, \hat{S}_z, \hat{\vec{S}}_A^2, \hat{\vec{S}}_B^2\}$

  $$|S, M_S; s_A, s_B\rangle$$

Both are complete orthonormal sets that span the space. One basis can be expanded in the other

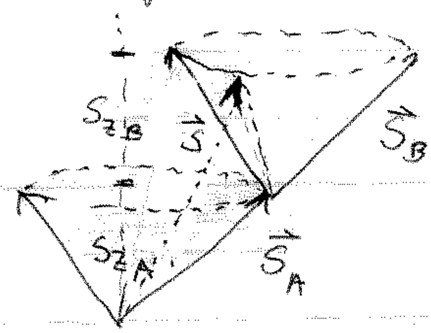
$$|S M_S; s_A, s_B\rangle = \sum_{m_{s_A}, m_{s_B}} |s_A m_{s_A}; s_B m_{s_B}\rangle \underbrace{\langle s_A m_{s_A}; s_B m_{s_B} | S M_S\rangle}$$

expansion coefficient

$\equiv$ Clebsch-Gordan coef.

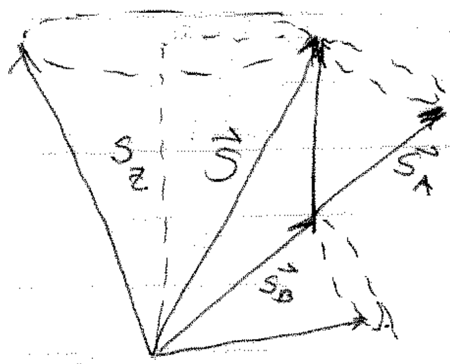
Vector picture

Uncoupled representation: "Good quantum numbers" are: magnitude of each spin and the z-projection of each spin. Pictorally



Note: $|\vec{S}|$ is uncertain

Coupled representation:
the "good quantum numbers" are:
the total magnitude of $\vec{S}$, the
and the individual z-projections



Note: $\hat{S}_{z_A}$ and $\hat{S}_{z_B}$ are uncertain