Physics 492: Quantum Mechanics II

Problem Set #1
Due: Tuesday, Jan. 29, 2019 (5:00pm in TA mailbox)

Problem 1: Unitary operators (20 Points)

An important class of operators are *unitary*, defined as those that *preserve inner product*, i.e. if $|\tilde{\psi}\rangle = \hat{U}|\psi\rangle$ and $|\tilde{\phi}\rangle = \hat{U}|\phi\rangle$, then $\langle\tilde{\phi}|\tilde{\psi}\rangle = \langle\phi|\psi\rangle$ and $\langle\tilde{\psi}|\tilde{\phi}\rangle = \langle\psi|\phi\rangle$.

(a) Show that unitary operators $\hat{U}^\dagger\hat{U} = \hat{U}\hat{U}^\dagger = \hat{1}$ (i.e. the adjoint is the inverse).

(b) Consider $\hat{U} = \exp\left(i\hat{A}\right)$, where $\hat{A}$ is a Hermitian operator. Show that $\hat{U}^\dagger = \exp\left(-i\hat{A}\right)$ and thus show $\hat{U}$ is unitary.

(c) Let $\hat{U}(t) = \exp\left(-i\hat{H}t/\hbar\right)$ where *t* is time and $\hat{H}$ is the Hamiltonian. Let $|\psi(0)\rangle$ be the state *t*=0. Show that $|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle$ satisfies the Time Dependent Schrödinger Equation. That is, the state evolves according to a unitary map. Explain why this is *required* by conservation of probability.

(d) Let $\{|u_n\rangle\}$ be the complete set of energy eigenfunctions, $\hat{H}|u_n\rangle = E_n|u_n\rangle$. Show that $\hat{U}(t) = \sum_n e^{-i\omega_n t}|u_n\rangle\langle u_n|$, where $\hbar\omega_n = E_n$. Using this show that $|\psi(t)\rangle = \sum_n c_n e^{-i\omega_n t}|u_n\rangle$, where $c_n = \langle u_n|\psi(0)\rangle$.

Problem 2: A two-dimensional Hilbert space (40 Points)

Consider a two dimensional Hilbert space spanned by an orthonormal basis $\{|\uparrow\rangle, |\downarrow\rangle\}$. Let us define the operators

$$\hat{S}_x = \frac{\hbar}{2}\left(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|\right),\ \hat{S}_y = \frac{\hbar}{2i}\left(|\uparrow\rangle\langle\downarrow| - |\downarrow\rangle\langle\uparrow|\right),\ \hat{S}_z = \frac{\hbar}{2}\left(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|\right).$$

(a) Show that each of these operators are Hermitian.
(b) Find the matrix representations of these operators in the basis $\{|\uparrow\rangle, |\downarrow\rangle\}$.
(c) Find the eigenvalues and eigenvectors of these operators.
(d) Show that, $\left[\hat{S}_x, \hat{S}_y\right] = i\hbar\hat{S}_z$ and cyclic permutations. Do this two way: Using the Dirac notation definition above and the matrix representations you found in (b).
(e) Let $|\pm\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle \pm |\downarrow\rangle\right)$. Find the matrix representations of these operators in the basis $\{|+\rangle, |-\rangle\}$; comment on your result.

(f) The matrices found in (b) and (e) are related through a *similarity transformation* by a unitary matrix, $\tilde{U}$,

$$\tilde{S}_x^{(\uparrow,\downarrow)} = \tilde{U}^\dagger \tilde{S}_x^{(\pm)} \tilde{U}, \quad \tilde{S}_y^{(\uparrow,\downarrow)} = \tilde{U}^\dagger \tilde{S}_y^{(\pm)} \tilde{U}, \quad \tilde{S}_z^{(\uparrow,\downarrow)} = \tilde{U}^\dagger \tilde{S}_z^{(\pm)} \tilde{U} ,$$

Here tilde denotes the *matrix representation* of the operator in the chosen basis indicated by the superscript.

Find the matrix $\tilde{U}$, show that it is unitary, and explicitly perform the three similarity transformations via matrix multiplication.

Now let $\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y$,

(g) Express $\hat{S}_\pm$ as outer products in the basis $\{|\uparrow\rangle, |\downarrow\rangle\}$, and as a matrix in this basis. Show that $\hat{S}_+^\dagger = \hat{S}_-$.

(h) Show that $\hat{S}_+|\uparrow\rangle = 0,\ \hat{S}_+|\downarrow\rangle = \hbar|\uparrow\rangle,\ \hat{S}_-|\uparrow\rangle = \hbar|\downarrow\rangle,\ \hat{S}_-|\downarrow\rangle = 0$, and find $\langle\uparrow|\hat{S}_+$, $\langle\downarrow|\hat{S}_+$, $\langle\uparrow|\hat{S}_-$, $\langle\downarrow|\hat{S}_-$.