Physics 492 - Quantum II

Problem Set #1 - Solutions

Problem 1: Unitary Operators

$\hat{U}$ preserves the inner product, i.e. $\forall|\psi\rangle$, $|\tilde{\psi}\rangle = \hat{U}|\psi\rangle$

$$\Rightarrow \langle\tilde{\phi}|\tilde{\psi}\rangle = \langle\phi|\psi\rangle$$

(a) But $\langle\tilde{\phi}|\tilde{\psi}\rangle = \langle\phi|\hat{U}^\dagger\hat{U}|\psi\rangle = \langle\phi|\psi\rangle$

$$\Rightarrow \boxed{\hat{U}^\dagger\hat{U} = \hat{\mathbb{1}}}$$

$$\Rightarrow \hat{U}^\dagger|\tilde{\psi}\rangle = \hat{U}^\dagger\hat{U}|\psi\rangle = \hat{\mathbb{1}}|\psi\rangle = |\psi\rangle$$

$$\therefore \langle\phi|\psi\rangle = \langle\tilde{\phi}|\hat{U}\hat{U}^\dagger|\tilde{\psi}\rangle = \langle\tilde{\phi}|\tilde{\psi}\rangle$$

$$\Rightarrow \boxed{\hat{U}\hat{U}^\dagger = \hat{\mathbb{1}}}$$

Thus for unitary operators, the inverse is the identity

(b) Let $\hat{U} = \exp(i\hat{A})$ where $\hat{A}$ is Hermitian (i.e. $\hat{A}^\dagger = \hat{A}$)

$$\hat{U} = \sum_{n=0}^{\infty}\frac{1}{n!}(i\hat{A})^n$$

$$\hat{U}^\dagger = \sum_{n=0}^{\infty}\frac{1}{n!}\left[(i\hat{A})^n\right]^\dagger = \sum_{n=0}^{\infty}\frac{1}{n!}\left[(-i\hat{A}^\dagger)\right]^n$$

$$\Rightarrow \hat{U}^\dagger = \sum_{n=0}^{\infty}\frac{1}{n!}(-i\hat{A})^n = \boxed{\exp(-i\hat{A})}$$

$$\text{thus}\quad \hat{U}^\dagger\hat{U} = e^{-i\hat{A}}\, e^{+i\hat{A}} = e^{-i\hat{A}+i\hat{A}} = e^{\overset{\text{zero operator}}{\hat{0}}}$$

$$= \hat{\mathbb{1}} = \hat{U}\hat{U}^\dagger \Rightarrow \hat{U} \text{ is unitary}$$

(Aside: <u>Note</u> in general $e^{\hat{A}} e^{\hat{B}} \neq e^{\hat{A}+\hat{B}}$ unless $\hat{A}$ and $\hat{B}$ <u>commute</u>)

(c) Let $\hat{U}(t) = e^{-i\hat{H}t/\hbar}$ where $\hat{H}$ = Hamiltonian

Given a state at $t=0$ $\quad |\psi(0)\rangle$

Let $|\psi(t)\rangle = \hat{U}(t)\,|\psi(0)\rangle$

$$\Rightarrow \frac{\partial}{\partial t}|\psi(t)\rangle = \frac{\partial \hat{U}(t)}{\partial t}|\psi(0)\rangle$$

(Aside $\quad \frac{\partial \hat{U}(t)}{\partial t} = \frac{\partial}{\partial t} e^{-i\hat{H}t/\hbar} = -\frac{i\hat{H}}{\hbar} e^{-i\hat{H}t/\hbar}$
$= -\frac{i}{\hbar}\hat{H}\,\hat{U}(t)$)

$$\Rightarrow \frac{\partial}{\partial t}|\psi(t)\rangle = -\frac{i\hat{H}}{\hbar}\hat{U}(t)|\psi(0)\rangle = -\frac{i\hat{H}}{\hbar}|\psi(t)\rangle$$

$$\Rightarrow \boxed{\frac{\hbar}{i}\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle}$$

thus $|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle$ is a solution to the Time Dependent Schrödinger Eqn

That $|\psi\rangle$ evolve according to unitary map is required by the interpretation of $|\psi\rangle$

At $|\psi(0)\rangle$ we choose $\||\psi(0)\rangle\|^2 = \langle\psi(0)|\psi(0)\rangle = 1$, normalized to a total probability of all possible alternatives. At a later time this total probability must be conserved

$$\Rightarrow \quad \||\psi(t)\rangle\|^2 = 1 \quad \Rightarrow \quad \langle\psi(t)|\psi(t)\rangle = \langle\psi(0)|\psi(0)\rangle$$

$$\Rightarrow \quad \underline{\text{Unitary time evolution}}$$

(d) Let $\{|u_n\rangle\}$ satisfy $\hat{H}|u_n\rangle = E_n|u_n\rangle$
complete set of orthonormal functions

$$\Rightarrow \quad \hat{\mathbb{1}} = \sum_n{}' |u_n\rangle\langle u_n|$$

$$\hat{U} = \hat{U} \sum_n{}' |u_n\rangle\langle u_n| = \sum_n{}' e^{-i\hat{H}t/\hbar} |u_n\rangle\langle u_n|$$

$$\Rightarrow \hat{U}(t) = \sum_n e^{-iE_nt/\hbar} |u_n\rangle\langle u_n| = \sum_n{}' e^{-i\omega_n t/\hbar} |u_n\rangle\langle u_n|$$

where $\hbar\omega_n = E_n$

Thus, $$|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle = \sum_n{}' e^{-i\omega_n t/\hbar} |u_n\rangle\underbrace{\langle u_n|\psi(0)\rangle}_{= C_n}$$

$$\Rightarrow \boxed{|\psi(t)\rangle = \sum_n{}' e^{-i\omega_n t/\hbar} C_n |u_n\rangle \quad \text{where } C_n = \langle u_n|\psi(0)\rangle}$$

Problem 2: A two-dimensional Hilbert Space

Orthonormal basis: $\{|\uparrow\rangle, |\downarrow\rangle\}$ $\langle\uparrow|\uparrow\rangle = \langle\downarrow|\downarrow\rangle = 1$
$\langle\uparrow|\downarrow\rangle = \langle\downarrow|\uparrow\rangle = 0$

$$\hat{S}_x = \frac{\hbar}{2}\left(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|\right)$$
$$\hat{S}_y = \frac{\hbar}{2i}\left(|\uparrow\rangle\langle\downarrow| - |\downarrow\rangle\langle\uparrow|\right)$$
$$\hat{S}_z = \frac{\hbar}{2}\left(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|\right)$$

(a) Recall $(|\psi\rangle\langle\phi|)^\dagger = |\phi\rangle\langle\psi|$

Thus, by inspection $\hat{S}_x^\dagger = \hat{S}_x$ ✓ and $\hat{S}_z^\dagger = \hat{S}_z$ ✓

$$\hat{S}_y^\dagger = \frac{\hbar}{-2i}\left(|\downarrow\rangle\langle\uparrow| - |\uparrow\rangle\langle\downarrow|\right) = \frac{\hbar}{2i}\left(|\uparrow\rangle\langle\downarrow| - |\downarrow\rangle\langle\uparrow|\right)$$
$$= \hat{S}_y \quad ✓ \qquad \text{all are Hermitian}$$

(b) Matrix representation in basis $\{|\uparrow\rangle, |\downarrow\rangle\}$

$$\hat{S}_x \doteq \begin{array}{cc} & \text{rows} \\ \begin{bmatrix} \langle\uparrow|\hat{S}_x|\uparrow\rangle & \langle\uparrow|\hat{S}_x|\downarrow\rangle \\ \langle\downarrow|\hat{S}_x|\uparrow\rangle & \langle\downarrow|\hat{S}_x|\downarrow\rangle \end{bmatrix} & \begin{matrix}\langle\uparrow| \\ \langle\downarrow|\end{matrix} \\ \text{column: } |\uparrow\rangle \quad\quad |\downarrow\rangle & \end{array}$$

$$\langle\uparrow|\hat{S}_x|\uparrow\rangle = \frac{\hbar}{2}\left(\langle\uparrow|\uparrow\rangle\cancelto{0}{\langle\downarrow|\uparrow\rangle} + \cancelto{0}{\langle\uparrow|\downarrow\rangle}\langle\uparrow|\uparrow\rangle\right) = 0$$

$$\langle\downarrow|\hat{S}_x|\downarrow\rangle = \frac{\hbar}{2}\left(\cancelto{0}{\langle\downarrow|\uparrow\rangle}\langle\downarrow|\downarrow\rangle + \langle\downarrow|\downarrow\rangle\cancelto{0}{\langle\uparrow|\downarrow\rangle}\right) = 0$$

$$\langle\uparrow|\hat{S}_x|\downarrow\rangle = \frac{\hbar}{2}\left(\underbrace{\langle\uparrow|\uparrow\rangle\langle\downarrow|\downarrow\rangle}_{=1} + \underbrace{\langle\uparrow|\downarrow\rangle}_{0}\underbrace{\langle\uparrow|\downarrow\rangle}_{0}\right) = 1$$

$$\langle\downarrow|\hat{S}_x|\uparrow\rangle = \langle\uparrow|\hat{S}_x|\downarrow\rangle^* \quad (\text{since } \hat{S}_x \text{ Hermitian})$$

$$= \frac{\hbar}{2}$$

$$\Rightarrow \boxed{\hat{S}_x \doteq \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$$

Similarly $\hat{S}_y \doteq \begin{bmatrix} \langle\uparrow|\hat{S}_y|\uparrow\rangle & \langle\uparrow|\hat{S}_y|\downarrow\rangle \\ \langle\downarrow|\hat{S}_y|\uparrow\rangle & \langle\downarrow|\hat{S}_y|\downarrow\rangle \end{bmatrix}$

$$\Rightarrow \boxed{\hat{S}_y \doteq \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}}$$

$$\hat{S}_z \doteq \begin{bmatrix} \langle\uparrow|\hat{S}_z|\uparrow\rangle & \langle\uparrow|\hat{S}_z|\downarrow\rangle \\ \langle\downarrow|\hat{S}_z|\uparrow\rangle & \langle\downarrow|\hat{S}_z|\downarrow\rangle \end{bmatrix}$$

$$\Rightarrow \boxed{\hat{S}_z \doteq \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}$$

(d) $[\hat{S}_x, \hat{S}_y] = \hat{S}_x\hat{S}_y - \hat{S}_y\hat{S}_x$

$$= \frac{\hbar^2}{4i}\Big[\big(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|\big)\big(|\uparrow\rangle\langle\downarrow| - |\downarrow\rangle\langle\uparrow|\big)$$

$$- \big(|\uparrow\rangle\langle\downarrow| - |\downarrow\rangle\langle\uparrow|\big)\big(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|\big)\Big]$$

$$= \frac{\hbar^2}{4i}\big(|\downarrow\rangle\langle\downarrow| - |\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow| - |\uparrow\rangle\langle\uparrow|\big)$$

Thus, $[\hat{S}_x, \hat{S}_y] = \frac{\hbar^2}{4i} \left(2|\downarrow\rangle\langle\downarrow| - 2|\uparrow\rangle\langle\uparrow| \right)$

$$= i\hbar \, \frac{\hbar}{2} \left(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow| \right)$$

$$\Rightarrow \boxed{[\hat{S}_x, \hat{S}_y] = i\hbar \hat{S}_z}$$

Cyclic permutation: $[\hat{S}_z, \hat{S}_x] = i\hbar \hat{S}_y$?

$$\hat{S}_z \hat{S}_x = \frac{\hbar^2}{4} \left(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow| \right) \left(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow| \right)$$

$$= \frac{\hbar^2}{4} \left(|\uparrow\rangle\langle\downarrow| - |\downarrow\rangle\langle\uparrow| \right)$$

$$\hat{S}_x \hat{S}_z = \frac{\hbar^2}{4} \left(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow| \right) \left(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow| \right)$$

$$= \frac{\hbar^2}{4} \left(|\downarrow\rangle\langle\uparrow| - |\uparrow\rangle\langle\downarrow| \right)$$

$$\Rightarrow \hat{S}_z \hat{S}_x - \hat{S}_x \hat{S}_z = \frac{\hbar^2}{2} \left(|\uparrow\rangle\langle\downarrow| - |\downarrow\rangle\langle\uparrow| \right)$$

$$= i\hbar \, \frac{\hbar}{2i} \left(|\uparrow\rangle\langle\downarrow| - |\downarrow\rangle\langle\uparrow| \right)$$

$$\Rightarrow \boxed{[\hat{S}_z, \hat{S}_x] = i\hbar \hat{S}_y}$$

Similarly, the same procedure yields

$$\boxed{[\hat{S}_y, \hat{S}_z] = i\hbar \hat{S}_x}$$

For comparison, let's carry out the same calculation using the matrix representations

$$\hat{S}_x \hat{S}_y \doteq \frac{\hbar^2}{4} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \frac{\hbar^2}{4} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$$\hat{S}_y \hat{S}_x \doteq \frac{\hbar^2}{4} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \frac{\hbar^2}{4} \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$$

$$\Rightarrow \hat{S}_x \hat{S}_y - \hat{S}_y \hat{S}_x \doteq \frac{\hbar^2}{2} \begin{bmatrix} +i & 0 \\ 0 & -i \end{bmatrix} = i\hbar \, \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow [\hat{S}_x, \hat{S}_y] \doteq i\hbar \hat{S}_z \quad \checkmark$$

We recognize these commutation relations as the same as those for the components of the <u>angular</u> <u>momentum</u> vector $\hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}}$

$$[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z \text{, and cycle perms.}$$

We will see that these operators represent the components of <u>spin</u> <u>angular</u> <u>momentum</u> for a spin-½ particle.

(c) We can find the eigenvalues and eigenvectors of these matrice by <u>diagonalization</u>

In the $\{|\uparrow\rangle, |\downarrow\rangle\}$ basis $\quad \hat{S}_x \doteq \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \hat{S}_y \doteq \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \hat{S}_z \doteq \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

- Thus, $\hat{S}_z$ is already diagonal, with eigenvectors $\{|\uparrow\rangle, |\downarrow\rangle\}$ corresponding to eigenvalues $\{+\frac{\hbar}{2}, -\frac{\hbar}{2}\}$

- Diagonalizing $\hat{S}_x$: $\det(\hat{S}_x - \lambda\hat{\mathbb{1}}) = \det\begin{bmatrix} -\lambda & \frac{\hbar}{2} \\ \frac{\hbar}{2} & -\lambda \end{bmatrix} = \lambda^2 - \left(\frac{\hbar}{2}\right)^2 = 0$

$\Rightarrow$ Eigenvalues $\lambda_\pm = \pm\frac{\hbar}{2}$ as for $\hat{S}_z$

The corresponding eigenvectors $|\pm\rangle_x = \alpha_\pm|\uparrow\rangle + \beta_\pm|\downarrow\rangle \doteq \begin{bmatrix} \alpha_\pm \\ \beta_\pm \end{bmatrix}$

$$\hat{S}_x|\pm\rangle_x \doteq \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \alpha_\pm \\ \beta_\pm \end{bmatrix} = \pm\frac{\hbar}{2}\begin{bmatrix} \alpha_\pm \\ \beta_\pm \end{bmatrix} \Rightarrow \frac{\hbar}{2}\beta_\pm = \pm\frac{\hbar}{2}\alpha_\pm \Rightarrow \beta_\pm = \pm\alpha_\pm$$

let $\alpha_\pm = 1 \Rightarrow \beta_\pm = \pm 1 \Rightarrow |\pm\rangle_x \doteq \begin{bmatrix} 1 \\ \pm 1 \end{bmatrix} \Rightarrow \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ \pm 1 \end{bmatrix}$ (normalized)

$$\Rightarrow \boxed{\text{Eigenvectors of } \hat{S}_x \quad |\pm\rangle_x = \frac{|\uparrow\rangle \pm |\downarrow\rangle}{\sqrt{2}} \quad \text{Eigenvalues } \left\{+\frac{\hbar}{2}, -\frac{\hbar}{2}\right\}}$$

- Diagonalizing $\hat{S}_y$: $\det(\hat{S}_y - \lambda\hat{\mathbb{1}}) = \det\begin{bmatrix} -\lambda & i\frac{\hbar}{2} \\ -i\frac{\hbar}{2} & -\lambda \end{bmatrix} = \lambda^2 - \left(\frac{\hbar}{2}\right)^2 = 0$

$\Rightarrow$ Eigenvalues $\lambda_\pm = \pm\frac{\hbar}{2}$ as for $\hat{S}_y$

The corresponding eigenvectors $|\pm\rangle_y = \gamma_\pm|\uparrow\rangle + \delta_\pm|\downarrow\rangle \doteq \begin{bmatrix} \gamma_\pm \\ \delta_\pm \end{bmatrix}$

$$\hat{S}_y|\pm\rangle_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} \gamma_\pm \\ \delta_\pm \end{bmatrix} = \pm\frac{\hbar}{2}\begin{bmatrix} \gamma_\pm \\ \delta_\pm \end{bmatrix} \Rightarrow -i\delta_\pm = \pm\gamma_\pm \Rightarrow \delta_\pm = \pm i\gamma_\pm$$

$$\delta_\pm = \pm i \Rightarrow |\pm\rangle_y \doteq \begin{bmatrix} 1 \\ \pm i \end{bmatrix} \Rightarrow |\pm\rangle_y \doteq \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ \pm i \end{bmatrix}$$

$$\Rightarrow \boxed{\text{Eigenvectors of } \hat{S}_y \quad |\pm\rangle_y = \frac{|\uparrow\rangle \pm i|\downarrow\rangle}{\sqrt{2}} \quad \text{Eigenvalues } \left\{+\frac{\hbar}{2}, -\frac{\hbar}{2}\right\}}$$

<u>Note:</u> the fact that $\hat{S}_x$, $\hat{S}_y$, and $\hat{S}_z$ all have the <u>same eigenvalues</u> means they are <u>"similar"</u> matrices. You can transform one matrix into another by a similarly (here unitary) matrix. This is a <u>rotation</u>.

(e) In this basis,

$$\hat{S}_x \doteq \begin{bmatrix} \langle + | \hat{S}_x | + \rangle & \langle + | \hat{S}_x | - \rangle \\ \langle - | \hat{S}_x | + \rangle & \langle - | \hat{S}_x | + \rangle \end{bmatrix}$$

$$\langle + | \hat{S}_x | + \rangle = \frac{\hbar}{2}\Big(\underbrace{\langle + | \uparrow \rangle}_{\frac{1}{\sqrt{2}}} \underbrace{\langle \downarrow | + \rangle}_{\frac{1}{\sqrt{2}}} + \underbrace{\langle + | \downarrow \rangle}_{\frac{1}{\sqrt{2}}} \underbrace{\langle \uparrow | + \rangle}_{\frac{1}{\sqrt{2}}} \Big) = \frac{\hbar}{2}$$

$$\langle + | \hat{S}_x | - \rangle = \frac{\hbar}{2}\Big(\underbrace{\langle + | \uparrow \rangle}_{+\frac{1}{\sqrt{2}}} \underbrace{\langle \downarrow | - \rangle}_{-\frac{1}{\sqrt{2}}} + \underbrace{\langle + | \downarrow \rangle}_{+\frac{1}{\sqrt{2}}} \underbrace{\langle \uparrow | - \rangle}_{+\frac{1}{\sqrt{2}}} \Big) = 0$$

$$= \langle - | \hat{S}_x | + \rangle^* \quad (\text{since } \hat{S}_x \text{ Hermitian})$$

$$\langle - | \hat{S}_x | - \rangle = \frac{\hbar}{2}\Big(\underbrace{\langle - | \uparrow \rangle}_{+\frac{1}{\sqrt{2}}} \underbrace{\langle \downarrow | - \rangle}_{-\frac{1}{\sqrt{2}}} + \underbrace{\langle - | \downarrow \rangle}_{-\frac{1}{\sqrt{2}}} \underbrace{\langle \uparrow | - \rangle}_{+\frac{1}{\sqrt{2}}} \Big) = -\frac{\hbar}{2}$$

$$\Rightarrow \boxed{\hat{S}_x^{(+,-)} \doteq \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}$$

$$\hat{S}_y \doteq \begin{bmatrix} \langle + | \hat{S}_y | + \rangle & \langle + | \hat{S}_y | - \rangle \\ \langle - | \hat{S}_y | + \rangle & \langle - | \hat{S}_y | - \rangle \end{bmatrix}$$

$$\langle + | \hat{S}_y | + \rangle = \frac{\hbar}{2i}\Big(\langle + | \uparrow \rangle \langle \downarrow | + \rangle - \langle + | \downarrow \rangle \langle \uparrow | + \rangle \Big) = 0$$

$$\langle + | \hat{S}_y | - \rangle = \frac{\hbar}{2i}\Big(\langle + | \uparrow \rangle \langle \downarrow | - \rangle - \langle + | \uparrow \rangle \langle \downarrow | - \rangle \Big) = -\frac{\hbar}{2i}$$

$$= \langle - | \hat{S}_y | + \rangle^*$$

$$\langle - | \hat{S}_y | - \rangle = \frac{\hbar}{2i}\left(\langle - | \uparrow \rangle \langle \downarrow | - \rangle - \langle - | \downarrow \rangle \langle \uparrow | - \rangle \right) = 0$$

$$\boxed{\hat{S}_y^{(+,-)} \doteq \frac{\hbar}{2}\begin{bmatrix} 0 & +i \\ -i & 0 \end{bmatrix}}$$

$$\hat{S}_z^{(+,-)} \doteq \begin{bmatrix} \langle + | \hat{S}_z | + \rangle & \langle + | \hat{S}_z | - \rangle \\ \langle - | \hat{S}_z | + \rangle & \langle - | \hat{S}_z | - \rangle \end{bmatrix}$$

$$\langle + | \hat{S}_z | + \rangle = \frac{\hbar}{2}\left(\underset{\frac{1}{2}}{|\langle + | \uparrow \rangle|^2} - \underset{\frac{1}{2}}{|\langle + | \downarrow \rangle|^2} \right) = 0$$

$$\langle + | \hat{S}_z | - \rangle = \frac{\hbar}{2}\left(\underbrace{\langle + | \uparrow \rangle \langle \uparrow | - \rangle}_{\frac{1}{2}} - \underbrace{\langle + | \downarrow \rangle \langle \downarrow | - \rangle}_{-\frac{1}{2}} \right) = \frac{\hbar}{2}$$

$$= \langle - | \hat{S}_z | + \rangle^*$$

$$\langle - | \hat{S}_z | - \rangle = \frac{\hbar}{2}\left(|\langle - | \uparrow \rangle|^2 - |\langle - | \downarrow \rangle|^2 \right) = 0$$

$$\Rightarrow \quad \boxed{\hat{S}_z^{(+,-)} \doteq \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$$

Note in this basis $\hat{S}_x$ is <u>diagonal</u>, whereas in the $\{|\uparrow\rangle, |\downarrow\rangle\}$ basis $\hat{S}_z$ is diagonal.

More on this later...

(f) Let $|e_1\rangle = |\uparrow\rangle \qquad |e_2\rangle = |\downarrow\rangle$

$|\tilde{e}_1\rangle = |+\rangle \qquad |\tilde{e}_2\rangle = |-\rangle$

Let $S_{ij} = \langle e_i | \hat{S} | e_j \rangle$ For any component $\hat{S}_x, \hat{S}_y, \hat{S}_z$

Matrix elements in $\{|\uparrow\rangle, |\downarrow\rangle\}$ basis

We want to relate these matrix elements to those in the $\{|+\rangle, |-\rangle\}$. To do this simply just insert a complete set

$$\sum_i |\tilde{e}_i\rangle\langle\tilde{e}_i| = \hat{\mathbb{1}}$$

$$S_{ij} = \langle e_i | \hat{\mathbb{1}} \hat{S} \hat{\mathbb{1}} e_j \rangle = \sum_{k,l} \langle e_i | \tilde{e}_k \rangle \langle \tilde{e}_k | \hat{S} | \tilde{e}_l \rangle \langle \tilde{e}_l | e_j \rangle$$

Now $\tilde{S}_{kl} \equiv \langle \tilde{e}_k | \hat{S} | \tilde{e}_l \rangle$ = matrix elements in $\{|+\rangle, |-\rangle\}$

Let $U_{lj} = \langle \tilde{e}_l | e_j \rangle$ these form the elements of a matrix

$$\hat{U} \doteq \begin{bmatrix} \langle \tilde{e}_1 | e_1 \rangle & \langle \tilde{e}_1 | e_2 \rangle \\ \langle \tilde{e}_2 | e_1 \rangle & \langle \tilde{e}_2 | \tilde{e}_2 \rangle \end{bmatrix} = \begin{bmatrix} \langle + | \uparrow \rangle & \langle + | \downarrow \rangle \\ \langle - | \uparrow \rangle & \langle - | \downarrow \rangle \end{bmatrix}$$

Note $\langle e_i | \tilde{e}_k \rangle = \langle \tilde{e}_k | e_i \rangle^* = (U_{ki})^* = U^\dagger_{ik}$

Thus $S_{ij} = \sum_{k,l} (U^{\dagger})_{ik} \; \tilde{S}_{kl} \; U_{lk}$

$$S = U^{\dagger} \tilde{S} U \quad \text{(matrix multiplication)}$$

Check: Is U unitary?

Using Dirac notation: $(U^{\dagger}U)_{ij} = \sum_k (U^{\dagger})_{ik} \; U_{kj}$

$$(U^{\dagger}U)_{ij} = \sum_k \langle e_i | \tilde{e}_k \rangle \langle \tilde{e}_k | e_j \rangle = \langle e_i | \underbrace{\sum_k |\tilde{e}_k\rangle\langle\tilde{e}_k|}_{= \mathbb{1}} | e_j \rangle$$

$$= \delta_{ij}$$

$$\Rightarrow \quad U^{\dagger}U = \mathbb{1} \quad \checkmark$$

Using matrices $\quad \hat{U} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$\hat{U}^{\dagger} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad (\hat{U}^{T} = \hat{U}$ and $\hat{U}$ real$)$

$$\hat{U}^{\dagger}\hat{U} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

Therefore U is unitary

$$\tilde{S}_x = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$U^+ \tilde{S}_x U = \frac{\hbar}{4}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{\hbar}{4}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{\hbar}{4}\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$$

$$\Rightarrow U^+ \tilde{S}_x U = S_x \quad \checkmark$$

$$\tilde{S}_y = \frac{\hbar}{2i}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$U^+ \tilde{S}_y U = \frac{\hbar}{4i}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{\hbar}{4i}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} = \frac{\hbar}{4i}\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$$

$$\Rightarrow U^+ \tilde{S}_y U = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \checkmark$$

$$\tilde{S}_z = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$U^+ \tilde{S}_z U = \frac{\hbar}{4}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{\hbar}{4}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{\hbar}{4}\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\Rightarrow U^+ \tilde{S}_z U = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \checkmark$$

Now let $\hat{S}_\pm = \dfrac{\hat{S}_x \pm i\,\hat{S}_y}{\hbar}$

(g) $\Rightarrow$ $\hat{S}_\pm = \frac{1}{2}(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|) \pm \frac{1}{2}(|\uparrow\rangle\langle\downarrow| - |\downarrow\rangle\langle\uparrow|)$

$\Rightarrow$ $\boxed{\hat{S}_+ = |\uparrow\rangle\langle\downarrow|}$ $\quad$ $\boxed{\hat{S}_- = |\downarrow\rangle\langle\uparrow|}$

By inspection $\hat{S}_+^\dagger = \hat{S}_-$

(h) $\hat{S}_+|\downarrow\rangle = |\uparrow\rangle \quad \hat{S}_+|\uparrow\rangle = 0$ (null vector)

$\hat{S}_-|\uparrow\rangle = |\downarrow\rangle \quad \hat{S}_-|\downarrow\rangle = 0$

thus $\hat{S}_\pm$ are "raising and lowering operators"

Finally, $\langle\uparrow|\hat{S}_+ = (\hat{S}_-|\uparrow\rangle)^\dagger = |\downarrow\rangle$

$\langle\downarrow|\hat{S}_+ = (\hat{S}_-|\downarrow\rangle)^\dagger = 0$

$\langle\uparrow|\hat{S}_- = (\hat{S}_+|\uparrow\rangle)^\dagger = 0$

$\langle\downarrow|\hat{S}_- = (\hat{S}_+|\downarrow\rangle)^\dagger = \langle\uparrow|$

Note $\hat{S}_+ \doteq \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \hat{S}_- \doteq \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

in this basis