Physics 492: Quantum Mechanics II

Problem Set #2
Due: Tuesday, Feb. 5, 2019 (5:00pm in TA mailbox)

Problem 1: Finish up Problem 2f from P.S. #1 (5 points)

Problem 2: Position Translation (25 points)

Consider the unitary operator $\hat{T}_{x_0} \equiv e^{-ix_0\hat{p}/\hbar}$.

(a) Let $|\psi\rangle$ be an arbitrary state with wave function (position representation) $\psi(x) = \langle x|\psi\rangle$. Now consider the state $|\psi'\rangle = \hat{T}_{x_0}|\psi\rangle$. Show that is wave function is

$$\psi'(x) = \langle x|\hat{T}_{x_0}|\psi\rangle = e^{-x_0\frac{d}{dx}}\psi(x) = \psi(x - x_0) \quad \text{(Hint: think about Taylor expansions)}$$

Use this to interpret why $\hat{T}_{x_0}$ is known as the "position translation operator."

(b) Consider the position eigenvectors, $|x\rangle$. Show that $\hat{T}_{x_0}|x\rangle = |x + x_0\rangle$. (Hint: insert a complete set of momentum eigenvectors).

(c) Use (b) to show that $\langle x|\hat{T}_{x_0} = \langle x - x_0|$, and use this to rederive (a)

(e) Show that $\hat{T}^{\dagger}_{x_0}\hat{x}\hat{T}_{x_0} = \hat{x} + x_0\hat{1}$ (Hint: express $\hat{x}$ is the position basis). Again, interpret.

(f) Extra credit (5 points). Let $x_0 = dx_0$, a *differential* length, and from part (e) we know that $\hat{T}^{\dagger}_{dx_0}\hat{x}\hat{T}_{dx_0} = \hat{x} + dx_0\hat{1}$ (differential translation). Use this to show that $[\hat{x}, \hat{p}] = i\hbar$. Cool!!

Problem 3: Center of mass and relative coordinates

Consider two particles of mass m_1 and m_2 moving on a line with position and momentum operators $\hat{x}_1, \hat{p}_1$ and $\hat{x}_2, \hat{p}_2$ respectively. Define

- $\hat{X}_{CoM} \equiv \dfrac{m_1\hat{x}_1 + m_2\hat{x}_2}{m_1 + m_2}$, $\hat{P}_{CoM} \equiv \hat{p}_1 + \hat{p}_2$ (center of mass position and momentum)
- $\hat{x}_{rel} \equiv \hat{x}_1 - \hat{x}_2$, $\hat{p}_{rel} \equiv \dfrac{m_2\hat{p}_1 - m_1\hat{p}_2}{m_1 + m_2}$ (relative position and momentum)

(a) Show that $\left[\hat{X}_{CoM}, \hat{P}_{CoM}\right] = \left[\hat{x}_{rel}, \hat{p}_{rel}\right] = i\hbar$, $\left[\hat{X}_{CoM}, \hat{p}_{rel}\right] = \left[\hat{x}_{rel}, \hat{P}_{CoM}\right] = 0$, that is the phase-space coordinates for a given degree of freedom satisfy the canonical commutation relation and those associated with different degrees of freedom commute.

(b) Suppose both particles are free particles. Show that the Hamiltonian is

$\hat{H} = \frac{\hat{p}_1^2}{2m_1} + \frac{\hat{p}_2^2}{2m_2} = \frac{\hat{P}_{CoM}^2}{2M} + \frac{\hat{p}_{rel}^2}{2\mu}$, where $M = m_1 + m_2$ is the total mass and $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is the so-called "reduced mass".

(c) We see in part (b) that the Hamiltonian is separable in both coordinate of particles 1 and 2 and also in center of mass coordinates. What symmetry explains this?

Now consider a two-particle state whose wave function in the position representation relative to coordinates x_1 and x_2 is $\Psi(x_1, x_2) = \delta(x_1 - x_2 - X_0)$.

(d) What is the two-particle momentum-space wave function $\tilde{\Phi}(p_1, p_2)$?

(e) Show that this state is a simultaneous eigenstate of $\hat{x}_{rel}$ and $\hat{P}_{CoM}$, and find the eigenvalues of these two operators. Is this consistent with the uncertainty principle?

This state is at the heart of the famous Einstein-Podolsky-Rosen (EPR) paradox and that first probed the mystery of entanglement, as we will study.