Physics 492: Quantum Mechanics II
Problem Set #2 Solutions

Problem 1: (See Problem Set #1 Solutions)

Problem 2: The Position Translation Operator

Consider $\hat{T}_{x_0} \equiv e^{-ix_0\hat{p}/\hbar}$

(a) If $\psi(x) = \langle x|\psi\rangle$ and $|\psi'\rangle \equiv \hat{T}_{x_0}|\psi\rangle$

$$\Rightarrow \psi'(x) = \langle x|\hat{T}_{x_0}|\psi\rangle = \langle x|e^{-ix_0\hat{p}/\hbar}|\psi\rangle$$

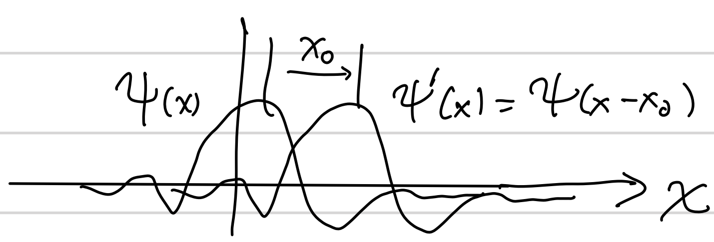
Recall, in position representation $\hat{p} \doteq \frac{\hbar}{i}\frac{\partial}{\partial x} \Rightarrow \psi'(x) = \exp\left\{\frac{-i}{\hbar}x_0\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\right\}\psi(x)$

$$\Rightarrow \psi'(x) = \underbrace{\exp\left\{-x_0\frac{d}{dx}\right\}}\psi(x) = \sum_{n=0}^{\infty}\frac{(-x_0)^n}{n!}\frac{d^n}{dx^n}\psi(x)$$

Differential operator (total derivative here since there is only one variable)

Recall $f(x+a) = \sum_{n=0}^{\infty}\frac{a^n}{n!}\frac{d^nf}{dx^n}$ (Taylor series expansion around $a=0$)

$\Rightarrow \boxed{\psi'(x) = \psi(x-x_0)}$ Thus $\hat{T}_{x_0}$ shifts (or "translates") $\psi(x)$ by x_0



(b) Consider a position eigenvector $|x\rangle$.

Then $\hat{T}_{x_0}|x\rangle = e^{\frac{-ix_0\hat{p}}{\hbar}}\,\hat{\mathbb{1}}\,|x\rangle = \int dp\, e^{\frac{-ix_0\hat{p}}{\hbar}}|p\rangle\langle p|x\rangle = \int dp\, e^{\frac{-ix_0 p}{\hbar}}|p\rangle\underbrace{\langle p|x\rangle}_{e^{-i\frac{px}{\hbar}}}$

eigenvalue

insert complete set of momentum eigenvectors

$$\Rightarrow \hat{T}_{x_0}|x\rangle = \int dp\, e^{\frac{ip}{\hbar}(x_0+x)}|p\rangle, \quad \text{But } |x+x_0\rangle = \int dp\,|p\rangle\langle p|x+x_0\rangle = \int dp\, e^{\frac{ip}{\hbar}(x+x_0)}|p\rangle$$

$$\Rightarrow \boxed{\hat{T}_{x_0}|x\rangle = |x+x_0\rangle}$$

(c) $\langle x|\hat{T}_{x_0} = (\hat{T}^{\dagger}_{x_0}|x\rangle)^{\dagger} = (e^{+\frac{i x_0 \hat{p}}{\hbar}}|x\rangle)^{\dagger} = (\hat{T}_{-x_0}|x\rangle)^{\dagger} = |x-x_0\rangle^{\dagger} = \langle x-x_0|$

Thus $\psi'(x) = \langle x|\hat{T}_{x_0}|\psi\rangle = \langle x-x_0|\psi\rangle = \psi(x-x_0)$ as in part (a)

(d) $\hat{T}^{\dagger}_{x_0}\hat{x}\hat{T}_{x_0} = \hat{T}^{\dagger}_{x_0}\underbrace{\int dx\, x\, |x\rangle\langle x|}_{\text{"Diagonal" representation of } \hat{x}}\hat{T}_{x_0} = \int dx\, x\, \hat{T}^{\dagger}_{x_0}|x\rangle\langle x|\hat{T}_{x_0} = \int dx\, x\, (\hat{T}_{-x_0}|x\rangle)(\langle x|\hat{T}_{x_0})$

$= \int dx\, x\, |x-x_0\rangle\langle x-x_0|$. Let $x' = x-x_0 \Rightarrow \hat{T}^{\dagger}_{x_0}\hat{x}\hat{T}_{x_0} = \int dx'\,(x'+x_0)|x'\rangle\langle x'|$

$= \underbrace{\int dx'\, x'|x'\rangle\langle x'|}_{\hat{x}} + x_0\underbrace{\int dx'|x'\rangle\langle x'|}_{\hat{1}} = \hat{x} + x_0\hat{1}$

Thus, the classical translation of coordinate $x \to x+x_0$ is implemented in quantum mechanics by a unitary transformation on the "observable" $\hat{T}^{\dagger}_{x_0}\hat{x}\hat{T}_{x_0} = \hat{x} + x_0\hat{1}$

(e) Now consider a "differential" translation by dx_0

$\hat{T}_{dx_0} = \exp\{\frac{-i}{\hbar}\hat{p}\,dx_0\} = \hat{1} - \frac{i}{\hbar}\hat{p}\,dx_0$ (differential is only first order)

$\hat{T}^{\dagger}_{dx_0}\hat{x}\hat{T}_{dx_0} = (\hat{1} - \frac{i}{\hbar}\hat{p}\,dx_0)^{\dagger}\hat{x}(\hat{1} - \frac{i}{\hbar}\hat{p}\,dx_0) = (\hat{1} + \frac{i}{\hbar}\hat{p}\,dx_0)\hat{x}(\hat{1} - \frac{i}{\hbar}\hat{p}\,dx_0)$

$= (\hat{x} + i\frac{\hat{p}\hat{x}}{\hbar}dx_0)(\hat{1} - \frac{i}{\hbar}\hat{p}\,dx_0) = \hat{x} - \frac{i}{\hbar}\hat{x}\hat{p}\,dx_0 + \frac{i}{\hbar}\hat{p}\hat{x}\,dx_0$ (to first order in dx_0)

$\Rightarrow \hat{T}^{\dagger}_{dx_0}\hat{x}\hat{T}_{dx_0} = \hat{x} - \frac{i}{\hbar}(\hat{x}\hat{p} - \hat{p}\hat{x})\,dx_0 = \hat{x} - \frac{i}{\hbar}[\hat{x},\hat{p}]\,dx_0$

But $\hat{T}^{\dagger}_{dx_0}\hat{x}\hat{T}_{dx_0} = \hat{x} + dx_0\hat{1} \Rightarrow \frac{-i}{\hbar}[\hat{x},\hat{p}] = \hat{1}$

$\Rightarrow \boxed{[\hat{x},\hat{p}] = i\hbar\,\hat{1}}$ The canonical commutator!

Upshot: This is an example of how commutation relations are expressions of symmetry. If a system is translationally invariant $\Rightarrow$ Constant potential $\Rightarrow$ no force $\Rightarrow$ momentum is conserved. Thus momentum is the generator of position translation. We have seen a similar pattern in problem Set #1. The Hamiltonian is the generator of time translation, because energy is conserved when the system is time translation invariant.