Physics 492: Quantum Mechanics II
Problem Set #4
Due: Thursday, Feb. 21, 2019 (5:00pm in TA mailbox)

Problem 1: The Isotropic Harmonic Oscillator in Two Dimensions (30 Points)

Consider a particle moving in a 2D isotropic harmonic potential $\hat{V}(x,y) = \frac{1}{2} m\omega^2 (\hat{x}^2 + \hat{y}^2)$.

(a) Show that the Hamiltonian commutes with $\hat{L}_z$. Please interpret.

(b) Defining the usual polar coordinates (ρ,ϕ) via $x = \rho\cos\phi$, $y = \rho\sin\phi$, show that the Hamiltonian can be written as

$$\hat{H} = \frac{1}{2m}\left(\hat{p}_\rho^2 + \frac{\hat{L}_z^2}{\rho^2}\right) + \frac{1}{2} m\omega^2 \hat{\rho}^2,$$

where $\hat{p}_\rho^2 = -\hbar^2 \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)$ is the radial momentum squared and $\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}$.

(c) Separating coordinates , writing the energy eigenstates $\psi_{n,m}(\rho,\phi) = R_{n,m}(\rho)\Phi_m(\phi)$, with $\Phi_m(\phi)$ the usual eigenstates of $\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}$, find the "radial equation" for $R_{n,m}(\rho)$.

(d) In class we solved this problem, separating in Cartesian coordinates $\Psi_{n_x,n_y}(x,y) = u_{n_x}(x)u_{n_y}(y)$ where $u_{n_x}(x)$ and $u_{n_y}(y)$ are the usual energy eigenfunctions in 1D. The energy eigenvalues are $E_n = \hbar\omega(n+1)$ with degeneracy $n+1$. It must be possible to expand these eigenfunction in terms of the eigenfunction in polar coordinates. Express the eigenfunctions $\Psi_{0,0}(x,y)$, $\Psi_{1,0}(x,y)$, $\Psi_{0,1}(x,y)$, and $\Psi_{1,1}(x,y)$ as

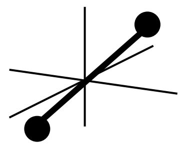
$$\Psi_{n_x,n_y}(x,y) = \sum_m c_{n,m}\psi_{n,m}(\rho,\phi), \quad n = n_x + n_y,$$

and thus find the eigenstates $\psi_{n,m}(\rho,\phi)$ in these expansions.

(e) Show that $R_{n,m}(\rho)$ satisfy the radial equation you found in (c)

(f) Finally, given the state $\Psi_{1,1}(x,y)$, what are the probabilities of finding the system with angular momentum eigenvalues $m = 2,1,0,-1,-2$ respectively?

Problem 2: The rigid rotator (20 points) -- POSTPONED
Consider a dumbbell model of a diatomic molecule, with two masses attached rigidly to a massless rod of length *d*.



(a) Assuming *d* cannot change, and its center of mass does not change, show that the Hamiltonian is

$$\hat{H} = \frac{\hat{L}^2}{2I^2}$$

where $\hat{L}^2$ is the squared angular momentum and *I* is the moment of inertia of the masses for rotation perpendicular to the dumbbell.
(b) What are the energy levels of the system? What is their degeneracy? What are the energy eigenfunctions? Sketch a level diagram.

(c) Diatomic nitrogen, with d=100 pm. Suppose a quantum jump occurs from level denoted by quantum number l+1 to l. What this the wavelength of the emitted?

(d) Suppose you measured the spectrum of emitted light between different transitions. Explain how you would you it to measure the moment of inertia of the molecule.

Problem 3: Uncertainty Principle for Angular Momentum Eigenstates (25 Points)

Consider a particle in an angular momentum eigenstate in the "standard basis," $|\psi\rangle = |l,m\rangle$

(a) Show that in this state $\left\langle \hat{L}_y \hat{L}_z \right\rangle = \left\langle \hat{L}_z \hat{L}_y \right\rangle = \hbar m \left\langle \hat{L}_y \right\rangle$ (Hint: Remember, $\hat{L}_z$ is a Hermitian operator).

(b) Use the commutator $\left[\hat{L}_y, \hat{L}_z\right]$ to show that $\left\langle \hat{L}_x \right\rangle = 0$.

(c) Follow a similar procedure to show that $\left\langle \hat{L}_y \right\rangle = 0$

(d) Assuming by symmetry $\left\langle \hat{L}_x^2 \right\rangle = \left\langle \hat{L}_y^2 \right\rangle$, Show that $\left\langle \hat{L}_x^2 \right\rangle = \left\langle \hat{L}_y^2 \right\rangle = \dfrac{\hbar^2\left(l(l+1) - m^2\right)}{2}$.
Hint: Consider $\hat{L}^2$.

(e) Show that this state obeys uncertainty relation, $\Delta L_x \Delta L_y \geq \frac{1}{2}\left|\left\langle \hat{L}_z \right\rangle\right|$.