Physics 492: Quantum Mechanics II
Problem Set # 4 Solutions

Problem 1: The 2D isotropic SHO

The Hamiltonian: $\hat{H} = \frac{\hat{p}_x^2 + \hat{p}_y^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$

We want to show that $[\hat{L}_z, \hat{H}] = 0$. There are many ways to do this. Here I will use operator algebra.

$$[\hat{L}_z, \hat{H}] = \frac{1}{2m}\left([\hat{L}_z,\hat{p}_x]\hat{p}_x + \hat{p}_x[\hat{L}_z,\hat{p}_x] + [\hat{L}_z,\hat{p}_y]\hat{p}_y + \hat{p}_y[\hat{L}_z,\hat{p}_y]\right)$$
$$+ \frac{1}{2}m\omega^2\left([\hat{L}_z,\hat{x}]\hat{x} + \hat{x}[\hat{L}_z,\hat{x}] + [\hat{L}_z,\hat{y}]\hat{y} + \hat{y}[\hat{L}_z,\hat{y}]\right)$$

Aside: $\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x \Rightarrow [\hat{L}_z, \hat{p}_x] = [\hat{x},\hat{p}_x]\hat{p}_y = i\hbar\hat{p}_y$

$[\hat{L}_z,\hat{p}_y] = [-\hat{y},\hat{p}_y]\hat{p}_x = -i\hbar\hat{p}_x$, $[\hat{L}_z,\hat{x}] = -\hat{y}[\hat{p}_x,\hat{x}] = i\hbar\hat{y}$, $[\hat{L}_z,\hat{y}] = \hat{x}[\hat{p}_y,\hat{y}] = -i\hbar\hat{x}$

$$\Rightarrow [\hat{L}_z, \hat{H}] = i\hbar\left(\frac{\hat{p}_y\hat{p}_x + \hat{p}_x\hat{p}_y - \hat{p}_x\hat{p}_y - \hat{p}_y\hat{p}_x}{2m}\right) + \frac{i\hbar m\omega^2}{2}(\hat{y}\hat{x} + \hat{x}\hat{y} - \hat{x}\hat{y} - \hat{y}\hat{x}) = 0 \checkmark$$

Note $[\hat{L}_z, \frac{\hat{P}^2}{2m}] = 0$. This is a statement of rotation symmetry of kinetic energy

$[\hat{L}_z, \hat{V}] = 0$ only if the potential has <u>azimuthal</u> symmetry

(b) In the position representations $\hat{H} \doteq \frac{-\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega^2(x^2+y^2)$

$$\nabla^2 \text{ is the Laplacian in 2D} = \underbrace{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}}_{\text{Cartesian}} = \underbrace{\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}}_{\text{Polar}}$$

The Cartesian and polar coordinates are related $x = \rho\cos\phi$, $y = \rho\sin\phi$

$$\Rightarrow \hat{H} = \frac{1}{2m}\left(\hat{P}_\rho^2 + \frac{\hat{L}_z^2}{\hat{\rho}^2}\right) + \frac{1}{2}m\omega^2\hat{\rho}^2$$

where $\hat{P}_\rho \doteq \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)$, $\hat{L}_z \doteq \frac{\hbar}{i}\frac{\partial}{\partial\phi}$

(c) Since $\hat{H}$ and $\hat{L}_z$ commute, we can separate the solution as

$$\Psi_{n,m}(\rho,\phi) = R_{n,m}(\rho)\,\Phi_m(\phi) \qquad \text{where} \quad \Phi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi} \quad \left(\text{eigenfunction of } \hat{L}_z\right)$$

The T.I.S.E. in 2D polar coordinates: nm

$$-\frac{\hbar^2}{2m}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial \Psi_{nm}}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Psi_{nm}(\rho,\phi)}{\partial\phi^2}\right) + \frac{1}{2}m\omega^2\rho^2 = E\,\Psi_{n,m}(\rho,\phi) = \hbar\omega(n+1)\,\Psi_{nm}$$

$$-\left(\frac{\Phi_m(\phi)}{\rho}\frac{d}{d\rho}\left(\rho\frac{dR_{nm}}{d\rho}\right) - \frac{m^2}{\rho^2}R_{nm}(\rho)\,\Phi_{nm}(\phi)\right) + \frac{1}{4}\underbrace{\left(\frac{2m\omega}{\hbar}\right)^2}_{\left(\frac{4}{\ell_c}\right)^2}\rho^2 = \underbrace{\left(\frac{2m\omega}{\hbar}\right)}_{\frac{4}{\ell_c^2}}(n+1)\,R_{nm}(\rho)\,\Phi_m(\phi)$$

$\frac{4}{\ell_c^2}$ – (Characteristic length) $\ell_c = \sqrt{\frac{2\hbar}{m\omega}}$

$$\frac{1}{\rho}\frac{dR_{nm}}{d\rho} + \frac{d^2R_{nm}}{d\rho^2} - \frac{m^2}{\rho^2}R_{nm} - 4\frac{\rho^2}{\ell_c^4} = -\frac{4}{\ell_c^2}(n+1)\,R_{nm}$$

The radial part of the wave function thus satisfies:

$$\Rightarrow \boxed{\begin{gathered}\bar{\rho}\,R'_{nm}(\bar{\rho}) + \bar{\rho}^2 R''_{nm}(\bar{\rho}) - 4\bar{\rho}^4 R_{nm}(\bar{\rho}) + 4(n+1)\bar{\rho}^2 R_{nm}(\bar{\rho}) - m^2 R_{n,m}(\bar{\rho}) = 0 \\ \text{where } \bar{\rho} \equiv \rho/\ell_c, \quad ' = \frac{d}{d\bar{\rho}}\end{gathered}}$$

(d) Separating in Cartesian coordinates, the energy eigenfunctions are

$$u_n(x) \propto \mathcal{H}_n(\sqrt{2}\,\bar{x})\, e^{-\bar{x}^2} \qquad \bar{x} = \frac{x}{\ell_c}$$

Hermite polynomial

$$u_0(x) \propto e^{-\bar{x}^2}, \quad u_1(\bar{x}) \propto \bar{x}\,e^{-\bar{x}^2}, \quad \Psi_{n_x,n_y}(x,y) \propto \mathcal{H}_{n_x}(\sqrt{2}\,\bar{x})\,\mathcal{H}_{n_y}(\sqrt{2}\,\bar{y})\,e^{-\bar{x}^2+\bar{y}^2}$$

$$\Rightarrow \Psi_{n_x=0,n_y=0}(x,y) \propto e^{-\bar{x}^2+\bar{y}^2}, \qquad \Psi_{n_x=1,n_y=0}(x,y) \propto \bar{x}\,e^{-\bar{x}^2+\bar{y}^2}$$

$$\Psi_{n_x=0,n_y=1}(x,y) \propto \bar{y}\,e^{-\bar{x}^2+\bar{y}^2}, \qquad \Psi_{n_x=1,n_y=1}(x,y) \propto \bar{x}\bar{y}\,e^{-\bar{x}^2+\bar{y}^2}$$

We to express these wavefunctions in cylindrical coordinates

$$\bar{x} = \bar{\rho}\cos\phi, \quad \bar{y} = \bar{\rho}\sin\phi$$

$\Psi_{n_x=0,n_y=0} \propto e^{-\bar{\rho}^2}$, $\Psi_{n_x=1,n_y=0} \propto \bar{\rho}\cos\phi\, e^{-\bar{\rho}^2}$, $\Psi_{n_x=0,n_y=1} \propto \bar{\rho}\sin\phi\, e^{-\bar{\rho}^2}$

$\Psi_{n_x=1,n_y=1} \propto \bar{\rho}^2 \sin\phi\cos\phi\, e^{-\bar{\rho}^2}$

Now, we know there are energy eigenstates that are also eigenstates of $\hat{L}_z$

$$|n,m\rangle \doteq \psi_{n,m}(\rho,\phi) \propto R_{nm}(\rho)\, e^{im\phi}$$

$|m=0\rangle \propto 1$, $\quad |m=\pm1\rangle \doteq e^{\pm im\phi}$, $\quad |m=\pm2\rangle \doteq e^{\pm i2\phi}$

Thus, we have two bases $\{|n_x,n_y\rangle\}$ and $\{|n,m\rangle\}$. We must be able expand elements of one basis in term of superposition of the others.

$$|n_x,n_y\rangle = \sum_{m=-n}^{n} c_{n,m}|n,m\rangle \quad \text{where} \quad n = n_x+n_y$$

$$\psi_{n_x n_y}(x,y) \propto \sum_{m=-n}^{n} c_{n,m} R_{nm}(\rho)\, e^{im\phi}$$

We will be able to "read off" the radial wave functions by "massaging" the wave functions into the form $R_{n,m}(\rho)\, e^{im\phi}$

$n_x=0, n_y=0 \quad \psi_{0,0} \propto e^{-\bar{\rho}^2} \propto R_{00}(\rho) \Rightarrow \boxed{R_{00}(\rho) \propto e^{-\bar{\rho}^2}} \quad n=0$

$\underbrace{n_x=1, n_y=0}_{n=1} \quad \psi_{1,0}(x,y) \propto \bar{\rho}\cos\phi\, e^{-\bar{\rho}^2/2} \propto \underbrace{\bar{\rho}e^{-\bar{\rho}^2/2} e^{i\phi}}_{|n=1,m=+1\rangle} + \underbrace{\bar{\rho}e^{-\bar{\rho}^2/2} e^{-i\phi}}_{|n=1,m=-1\rangle}$ (using $\cos\phi \propto e^{i\phi}+e^{-i\phi}$)

$$\Rightarrow \boxed{R_{1,\pm1}(\bar{\rho}) = \bar{\rho}e^{-\bar{\rho}^2}}$$

$\underbrace{n_x=0, n_y=1}_{n=1} \quad \psi_{1,0}(x,y) \propto \bar{\rho}\sin\phi\, e^{-\bar{\rho}^2} \propto \underbrace{-i\bar{\rho}e^{-\bar{\rho}^2} e^{-i\phi}}_{-i|n=1,m=+1\rangle} + \underbrace{i\bar{\rho}\, e^{-\bar{\rho}^2} e^{i\phi}}_{i|n=1,m=+1\rangle}$ (using $\sin\phi \propto \frac{e^{i\phi}-e^{-i\phi}}{i}$)

this agrees with what we found in class

$$|n=1, m=\pm1\rangle \propto |n_x=1,0\rangle \pm i|n_x=0,n_y=1\rangle$$

Finally, $\psi_{n_x=1,n_y=1}(\rho,\phi) \propto \bar\rho^2 \sin\phi\cos\phi \, e^{-\bar\rho^2} \propto \bar\rho^2 \sin 2\phi \, e^{-\bar\rho^2}$ $\qquad n=2$

$$\propto \underbrace{-i\bar\rho^2 e^{-\bar\rho^2} e^{2i\phi}}_{|2,+2\rangle} + \underbrace{i\bar\rho^2 e^{-\bar\rho^2} e^{-2i\phi}}_{|2,-2\rangle}$$

$$\Rightarrow \boxed{R_{2,\pm2}(\bar\rho) \propto \bar\rho^2 e^{-\bar\rho^2}}$$

(e) Check that our solutions satisfy the radial equation

$$\bar\rho R'_{nm}(\bar\rho) + \bar\rho^2 R''_{nm}(\bar\rho) - 4\bar\rho^4 R_{nm}(\bar\rho) + 4(n+1)\bar\rho^2 R_{nm}(\bar\rho) - m^2 R_{n,m}(\bar\rho) = 0$$

<u>$n=0, m=0$:</u> $\Rightarrow$ $\bar\rho R'_{00} + \bar\rho^2 R''_{00} - 4\bar\rho^4 R_{00} + 4\bar\rho^2 R_{00} \stackrel{?}{=} 0$

$R_{00}(\bar\rho) = e^{-\bar\rho^2}$ (Normalization unimportant)

$$[\bar\rho(-2\bar\rho e^{-\bar\rho^2}) + \bar\rho^2(-2e^{-\bar\rho^2} + 4\bar\rho^2 e^{-\bar\rho^2}) - 4\bar\rho^4 e^{-\bar\rho^2} + 4\bar\rho^2 e^{-\bar\rho^2} \stackrel{?}{=} 0 \quad ✓$$

<u>$n=1, m=\pm1$</u> $\Rightarrow$ $\bar\rho R'_{1,\pm1} + \bar\rho^2 R''_{1,\pm1}(\bar\rho) - 4\bar\rho^4 R_{1,\pm1}(\bar\rho) + 8\bar\rho^2 R_{1,\pm1}(\bar\rho) - R_{1,\pm}(\bar\rho) \stackrel{?}{=} 0$

$R_{1,\pm1}(\bar\rho) = \bar\rho e^{-\bar\rho^2}$ (unnormalized)

$$\bar\rho(1-2\bar\rho^2)e^{-\bar\rho^2} + \bar\rho^2(-4\bar\rho e^{-\bar\rho^2} - 2\bar\rho(1-2\bar\rho^2)e^{-\bar\rho^2}) - 4\bar\rho^5 e^{-\bar\rho^2} + 8\bar\rho^3 R_{1,\pm1} - R_{1,\pm1} \stackrel{?}{=} 0$$

$$\bar\rho - 2\bar\rho^3 - 6\bar\rho^3 + 4\bar\rho^5 - 4\bar\rho^5 + 8\bar\rho^3 - \bar\rho = 0 \quad ✓$$

<u>$n=2, m=\pm2$</u> $\Rightarrow$ $\bar\rho R'_{2,\pm2} + \bar\rho^2 R''_{2,\pm2}(\bar\rho) - 4\bar\rho^4 R_{2,\pm2}(\bar\rho) + 12\bar\rho^2 R_{2,\pm2}(\bar\rho) - 4R_{2,\pm2}(\bar\rho) \stackrel{?}{=} 0$

$R_{2,\pm2} = \bar\rho^2 e^{-\bar\rho^2}$ (unnormalized)

$$\Rightarrow \bar\rho(2\bar\rho - 2\bar\rho^3)e^{-\bar\rho^2} + \bar\rho^2(2 - 6\bar\rho^2 - 2\bar\rho(2\bar\rho - 2\bar\rho^3))e^{-\bar\rho^2} - 4\bar\rho^6 e^{-\bar\rho^2} + 12\bar\rho^4 e^{-\bar\rho^2} - 4\bar\rho^2 e^{-\bar\rho^2} \stackrel{?}{=} 0$$

$$2\bar\rho^2 - 2\bar\rho^4 + 2\bar\rho^2 - 6\bar\rho^4 - 4\bar\rho^4 + 4\bar\rho^6 - 4\bar\rho^6 + 12\bar\rho^4 - 4\bar\rho^2 \stackrel{?}{=} 0 \quad ✓$$

(f) We see by inspection that $\psi_{n_x=1,n_y=1}$ is an equal superposition of $m=\pm2$

$\Rightarrow$ Prob of finding $m=+2$ = Prob of $m=-2$ $= \frac{1}{2}$, all other prob $= 0$

Problem 3: Uncertainty Principle for Angular Momentum States

Given state $|\psi\rangle = |l,m\rangle$

(a) $\langle \hat{L}_y \hat{L}_z \rangle = \langle l,m|\hat{L}_y \overbrace{\hat{L}_z|l,m\rangle}^{\text{eigenstate}} = \hbar m \langle l,m|\hat{L}_y|l,m\rangle = \hbar m \langle \hat{L}_y \rangle$

$\langle \hat{L}_z \hat{L}_y \rangle = \underbrace{\langle l,m|\hat{L}_z}_{\text{eigenstate of Hermitian operator}} \hat{L}_y|l,m\rangle = \hbar m \langle lm|\hat{L}_y|lm\rangle = \hbar m \langle \hat{L}_y \rangle$

(b) $\langle l,m|[\hat{L}_y,\hat{L}_z]|lm\rangle = \underbrace{\langle lm|(\hat{L}_y\hat{L}_z - \hat{L}_z\hat{L}_y)|l,m\rangle}_{=0 \text{ from part (a)}} = \langle lm|i\hbar\hat{L}_x|lm\rangle$

$\Rightarrow \langle l,m|\hat{L}_x|l,m\rangle = 0$

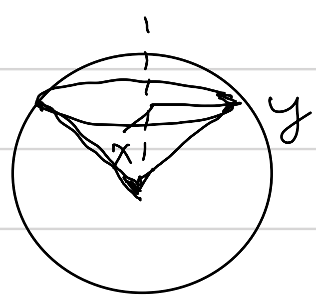
(c) Similarly $\langle l,m|[\hat{L}_x,\hat{L}_z]|lm\rangle = \underbrace{\langle l,m|\hat{L}_x\hat{L}_z - \hat{L}_z\hat{L}_x|l,m\rangle}_{=0 \text{ from same argument as part (a)}} = \langle l,m|-i\hbar\hat{L}_y|lm\rangle$

$\Rightarrow \langle l,m|\hat{L}_y|l,m\rangle = 0$

(d) Since $|l,m\rangle$ is an eigenstate of $\hat{L}_z$ is unchanged by a rotation about the z-axis

$\Rightarrow$ Any expectation of and operator along x or y must be equal.

This make sense for the "vector picture" of angular momentum



azimuthally symmetry

Now $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 \Rightarrow \langle l,m|\hat{L}^2|l,m\rangle = \hbar^2 l(l+1) = \langle \hat{L}_x^2\rangle + \langle \hat{L}_y^2\rangle + \langle \hat{L}_z^2\rangle$

$\Rightarrow 2\langle \hat{L}_x^2\rangle + (\hbar m)^2 = 2\langle \hat{L}_y^2\rangle + (\hbar m)^2 = \hbar^2 l(l+1)$

$\Rightarrow \langle \hat{L}_x^2\rangle = \langle \hat{L}_y^2\rangle = \hbar^2\left(\frac{l(l+1) - m^2}{2}\right)$

(e) The generalized uncertainty principle $\Delta L_x \Delta L_y \geq \frac{\hbar}{2}|\langle \hat{L}_z\rangle|$

For the "standard basis" states $\langle \hat{L}_z\rangle = \hbar m$ $\quad \langle \hat{L}_x\rangle = \langle \hat{L}_y\rangle = 0$

$\Rightarrow \sqrt{\langle \hat{L}_x^2\rangle\langle \hat{L}_y^2\rangle} = \hbar^2\left(\frac{l(l+1) - m^2}{2}\right) \geq \frac{\hbar^2 m}{2}$. This is true since $m^2 \leq l^2$

Note when $l = \pm m$ $\quad$ L.H.S $= \frac{\hbar^2 l}{2}$, $\quad$ R.H.S $= \frac{\hbar^2 l}{2}$:

Minimum Uncertainty State.