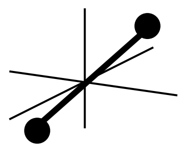
Physics 492: Quantum Mechanics II
Problem Set #4
Due: Tuesday, Mar. 19, 2019 (5:00pm in TA mailbox)

Problem 1: The rigid rotator (30 points)
Consider a dumbbell model of a diatomic molecule, with two masses attached rigidly to a massless rod of length *d*.



(a) Assuming *d* cannot change, and its center of mass does not change, show that the Hamiltonian is

$$\hat{H} = \frac{\hat{L}^2}{2I}$$

where $\hat{L}^2$ is the squared angular momentum and *I* is the moment of inertia of the masses for rotation perpendicular to the dumbbell.

(b) What are the energy levels of the system? What is their degeneracy? What are the energy eigenfunctions? Sketch a level diagram.

(c) Diatomic nitrogen, with d=100 pm. Suppose a quantum jump occurs from level denoted by quantum number l+1 to l. What this the wavelength of the emitted?

(d) Suppose you measured the spectrum of emitted light between different transitions. Explain how you would you it to measure the moment of inertia of the molecule.

Problem 2: Exercises with the hydrogen atom (from the textbook by Liboff) (**30 points**)

Consider a hydrogen atom where the electron motion relative to the proton is prepared in an initial state whose wave function (position representation) is given by

$$\psi(\mathbf{r},0) = \frac{4}{(2a_0)^{3/2}}\left[e^{-r/a_0} + A\frac{r}{a_0}e^{-r/2a_0}\left(-iY_{1,1}(\theta,\phi) + Y_{1,-1}(\theta,\phi) + \sqrt{7}Y_{1,0}(\theta,\phi)\right)\right]$$

(a) Write this as a ket that is a superposition of energy eigenstates $|n,l,m\rangle$, and in doing so find the constant *A* so the state is normalized.

(b) If one were to perform and projective measurement of the observable $\hat{L}^2$, what are the possible measurement outcomes, and with what probability do they occur?

(c) What is the probability density $P(r)$ the electron will be found in a spherical shell between r and $r+dr$?

(d) At what value of r is $P(r)$ maximum?

(e) If the atom is isolated from all other interactions, what is the wave function at a later time?

(f) If at t=0 one does a projective measurement of $\hat{L}_z$ on the atom prepared in $\psi(\mathbf{r},0)$ and one finds the outcome $+\hbar$, and then the hydrogen atom is isolated from all other interactions for t>0, what is the wave function at a later time?

(g) Repeat (f) but now one finds the outcome 0 for the $\hat{L}_z$ measurement.

Problem 3: Dipole Matrix Elements in Hydrogen. (30 points)
(a) Show that parity commutes with any component of angular momentum $\left[\hat{\Pi},\hat{L}_i\right]=0$, and $\left[\hat{\Pi},\hat{L}^2\right]=0$. (Hint: First show $\hat{\Pi}^\dagger\hat{L}_i\hat{\Pi}=\hat{L}_i$).

Thus the "standard basis" of angular momentum are also parity eigenvectors. One can show (you need not do so) that $\hat{\Pi}|l,m\rangle=(-1)^l|l,m\rangle$.

Now consider the electric dipole operator for the hydrogen atom, $\hat{\mathbf{d}}=-e\hat{\mathbf{r}}$, where $\hat{\mathbf{r}}$ is the position of the electron relative to the proton.

(b) Show that parity conservation implies that the matrix element with respect to the hydrogen energy eigenstates, $\langle n',l',m'|\hat{\mathbf{d}}|n,l,m\rangle$, vanishes exactly unless $(-1)^{l'-l}=-1$. Thus the expected value of the electric dipole in any energy state $|n,l,m\rangle$ is zero.

(c) Now consider the matrix of the z-component of the electric dipole, of between the 1s state and the three $2p$ states $\langle 2,1,m'|\hat{d}_z|1,0,m\rangle=-e\langle 2,1,m'|\hat{z}|1,0,m\rangle$. Evaluate the matrix element for all possible values of m,m'. (*Hint*, use spherical coordinates and the orthogonality of the spherical harmonics).