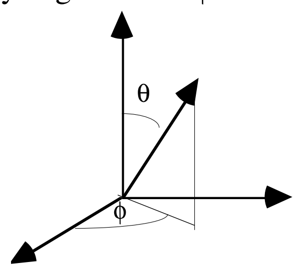
Physics 492: Quantum Mechanics II
Problem Set #6
Due: Thursday, March 28, 2019 (@5pm in TA mailbox)

Problem 1: Spin-1/2 along an arbitrary direction (20 points)

Given a unit vector $\vec{\mathbf{e}}_n$, defined by angles θ and ϕ with respect to the polar axis *z*,



(a) Show that the state $|\uparrow_n\rangle = \cos(\theta/2)|\uparrow_z\rangle + e^{i\phi}\sin(\theta/2)|\downarrow_z\rangle$, where $\{|\uparrow_z\rangle, |\downarrow_z\rangle\}$ is the standard basis, is an eigenstate of $\hat{\sigma}_n \equiv \vec{\mathbf{e}}_n \cdot \hat{\vec{\sigma}}$ with eigenvalue 1, $\hat{\sigma}_n|\uparrow_n\rangle = |\uparrow_n\rangle$.
Thus we interpret $|\uparrow_n\rangle$ as "spin-up along the direction $\vec{\mathbf{e}}_n$".

(b) Show that $|\uparrow_{-n}\rangle = |\downarrow_n\rangle$. Check for $\vec{\mathbf{e}}_n = \vec{\mathbf{e}}_z$.

(c) Use (a) and (b) to express $\{|\uparrow_x\rangle, |\downarrow_x\rangle, |\uparrow_y\rangle, |\downarrow_y\rangle\}$ in the standard basis.

(d) An aribitray (pure) state of a spin-1/2 particle can be express in the standard basis as

$$|\psi\rangle = \alpha|\uparrow_z\rangle + \beta|\downarrow_z\rangle$$

Show that up to an overall (irrelevant) phase $|\psi\rangle \equiv |\uparrow_n\rangle$ and find the (θ,ϕ) coordinates that define the *n* in terms of α and β.
Thus any (pure) state of a spin-1/2 is "spin up" along some direction.

(e) **Extra Credit:** Show that the inner product between any two pure states is, $\left|\langle\uparrow_n|\uparrow_{n'}\rangle\right|^2 = \frac{1+\vec{\mathbf{e}}_n \cdot \vec{\mathbf{e}}_{n'}}{2} = \cos^2(\Theta/2)$, where Θ is the angle between the directions $\vec{\mathbf{e}}_n$ and $\vec{\mathbf{e}}_{n'}$ in three dimensional space. Is this consistent with the statement "spin-up and spin-down are orthogonal states"?

Problem 2: The Rotation Operator for spin-1/2 (15 points)

We have learned that the operator $R_n(\Theta) = \exp\left\{-i\Theta(\vec{\mathbf{e}}_n \cdot \hat{\mathbf{J}})/\hbar\right\}$ is a "rotation operator", which rotates a vector about the axis $\vec{\mathbf{e}}_n$ by an angle Θ. For the case of spin 1/2,

$$\hat{\mathbf{J}} = \hat{\mathbf{S}} = \frac{\hbar}{2}\hat{\vec{\sigma}} \Rightarrow R_n(\Theta) = \exp\{-i\Theta\hat{\sigma}_n/2\}.$$

(a) Show that for spin 1/2, $R_n(\Theta) = \cos\left(\frac{\Theta}{2}\right)\hat{1} - i\sin\left(\frac{\Theta}{2}\right)\hat{\sigma}_n$. (Hint: Expand exponential)

(b) Show: $R_n(\Theta = 2\pi) = -\hat{1}$ -- Comment.

(c) Consider a series of rotations: Rotate about the y-axis by θ followed by a rotation about the z-axis by ϕ. Convince yourself that this takes the unit vector along $\vec{\mathbf{e}}_z$ to $\vec{\mathbf{e}}_n$. Show that up to an overall phase, $|\uparrow_n\rangle = R_z(\phi)R_y(\theta)|\uparrow_z\rangle$.