Physics 492: Quantum Mechanics II
Problem Set #7
Due: Thursday, April 4, 2019 (@ 5PM in TA mailbox)

Problem 1: Qubits (25 points)
A bit of information can be encoded into a spin-1/2 particle by, for example associating "logical zero" with spin-up and "logical one" with spin-down in the standard basis. Thus we define the "computation basis" $|0\rangle = |\uparrow_z\rangle$, $|1\rangle = |\downarrow_z\rangle$. A measurement of $\hat{\sigma}_z$ yields a bit of information, $|0\rangle$ or $|1\rangle$. Things are much more interesting than this – we can have *superpositions* of $|0\rangle$ and $|1\rangle$! Thus, the most general (pure) state of a *quantum bit* or ***qubit*** is $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. Quantum information is processed with qubits.

(a) Draw a Bloch sphere denoting the pure states of the qubit along the x,y,z axes.

We process classical bits with "logic" gates. For a single bit, the only reversible gates are the Identity and the NOT gates which obey the following truth tables:

in	out
0	0
1	1

Identity

in	out
0	1
1	0

NOT

A *quantum logic gate* is a unitary map of a qubit and thus a rotation on the Bolch sphere. The identity map is the identity matrix $\hat{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ in the computational basis. When dealing with quantum information we usually simplify the notation and denote the Pauli matrices as $X = \hat{\sigma}_x$, $Y = \hat{\sigma}_y$, $Z = \hat{\sigma}_z$ in the computational basis (here we drop the hats; operators are understood).

(b) Show that the "NOT gate" is $NOT = X$, in that it has the appropriate action on the computational basis states. Show that NOT can be implemented as a rotation about the x-axis on the Bloch sphere by π (remember, the overall phase of the state is irrelevant).

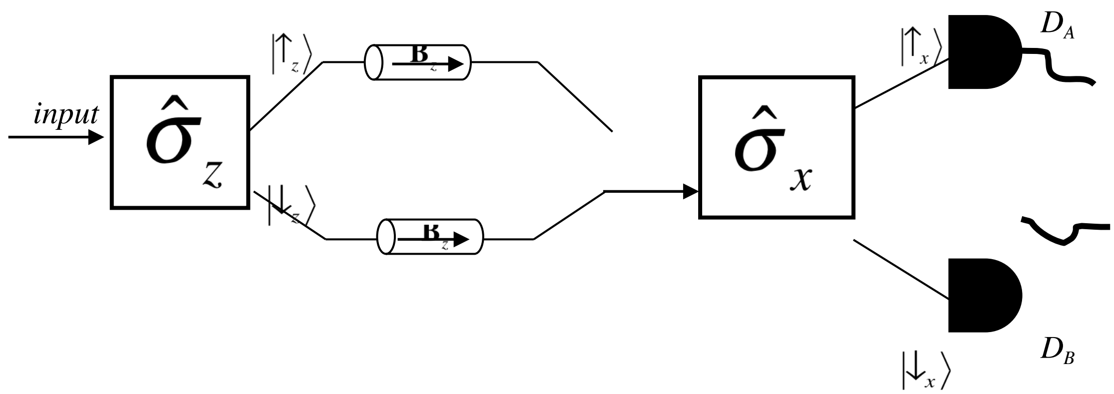
(c) We have entirely new logic gates on qubits that have no classical analog. For example we can define $\sqrt{NOT}$ such that $\sqrt{NOT}\sqrt{NOT} = NOT$! In other words $\sqrt{NOT}$ is a rotation about the x-axis by $\pi/2$. Write a matrix representation of $\sqrt{NOT}$ in the computational basis. What is are output states when $\sqrt{NOT}$ acts on $|0\rangle$ and when it acts on $|1\rangle$?

(d) The Hadamard gate is defined $H = \dfrac{X+Z}{\sqrt{2}}$. Write a matrix representation of H is the computational basis. What is are output states when H acts on $|0\rangle$ and when it acts on $|1\rangle$?

(e) Show that $H^\dagger XH = Z$, $H^\dagger ZH = X$, i.e. H flips X and Z.

(f) (Extra Credit – 5 points) Show that H can be implemented as rotation on the Bloch sphere. What is the axis and angle of rotation?

Problem 2: Mixed states vs. pure states and interference (25 points)
A "spin-interferometer" is shown below



Spin-1/2 electrons are prepared in a given state (pure or mixed) are separated in two paths by a Stern-Gerlach apparatus (gradient field along z). In each path the particle passes through a solenoid, with a uniform magnetic field along the z-axis. The two paths are then recombined, sent through another Stern-Gerlach with gradient along x, and the particles are counted in detectors in the two emerging ports.

The strength of the magnetic field is chosen so that $\Omega t = \phi$, for some phase ϕ, where $\Omega = 2\mu_B B/\hbar$ is the Larmor frequency and t is the time spent inside the solenoid.

(a) If the imput state is a pure state $|\psi_{in}\rangle = c_\uparrow|\uparrow_z\rangle + c_\downarrow|\downarrow_z\rangle$, what is the state state at the input to the $\hat{\sigma}_x$ -Stern Gerlach aparatus?

(b) Plot the probability of electrons arriving at detector D_B as a function of ϕ for the following pure state inputs: $(i)\,|\uparrow_z\rangle$, $(ii)\,|\uparrow_x\rangle$, $(iii)\,|\downarrow_x\rangle$.

(c) Repeat part (b) for the following mixed state inputs

$(i)\ \hat{\rho} = \frac{1}{2}|\uparrow_z\rangle\langle\uparrow_z| + \frac{1}{2}|\downarrow_z\rangle\langle\downarrow_z|$, $(ii)\ \hat{\rho} = \frac{1}{2}|\uparrow_x\rangle\langle\uparrow_x| + \frac{1}{2}|\downarrow_x\rangle\langle\downarrow_x|$, $(iii)\ \hat{\rho} = \frac{1}{3}|\uparrow_z\rangle\langle\uparrow_z| + \frac{2}{3}|\downarrow_z\rangle\langle\downarrow_z|$,

$(iv)\ \hat{\rho} = \frac{1}{3}|\uparrow_x\rangle\langle\uparrow_x| + \frac{2}{3}|\downarrow_x\rangle\langle\downarrow_x|$

Comment on your results.