**Physics 492: Quantum Mechanics II**
**Problem Set #7**
**Due: Thursday, April 11, 2019 (@ 5PM in TA mailbox)**

**Problem 1: Rabi Oscillations (30 points)**
We return here to the magnetic resonance problem studied in class. A spin (say an electron) is placed in a strong magnetic field along the $z$-axis, $\mathbf{B}_0 = B_{\parallel}\mathbf{e}_z$. In addition, a weak transverse field rotates in the $x$-y plane, $\mathbf{B}_1 = B_{\perp}\left(\cos\omega t\,\mathbf{e}_x + \sin\omega t\,\mathbf{e}_y\right)$.

(a) Show that one can write the Hamiltonian as

$$\hat{H}(t) = \frac{\hbar\Omega_{\parallel}}{2}\hat{\sigma}_z + \frac{\hbar\Omega_{\perp}}{2}\left(\hat{\sigma}_+ e^{-i\omega t} + \hat{\sigma}_- e^{+i\omega t}\right)$$

where $\hbar\Omega_{\parallel} = 2\mu_B B_{\parallel}$, $\hbar\Omega_{\perp} = 2\mu_B B_{\perp}$.

(b) For such a time-dependent Hamiltonian, we take as an Ansatz for the state $\left|\psi(t)\right\rangle = c_{\uparrow}(t)e^{-\frac{i\Omega_{\parallel}t}{2}}\left|\uparrow\right\rangle + c_{\downarrow}(t)e^{\frac{+i\Omega_{\parallel}t}{2}}\left|\downarrow\right\rangle$. The explicit time dependence stems from the "free evolution" under the strong field. The slower perturbed evolution is accounted for by the complex amplitudes. Use the time dependent Schrödinger equation to show that

$$\dot{c}_{\uparrow}(t) = -i\frac{\Omega_{\perp}}{2}c_{\downarrow}(t)e^{-i\Delta t}$$
$$\dot{c}_{\downarrow}(t) = -i\frac{\Omega_{\perp}}{2}c_{\uparrow}(t)e^{+i\Delta t}$$

where is $\Delta = \omega - \Omega_{\parallel}$ is the "detuning" of the perturbing frequency from resonance.

(c) Find the solution to these equations on resonance, $\Delta = 0$. The answer you should find is

$$c_{\uparrow}(t) = c_{\uparrow}(0)\cos\left(\frac{\Omega_{\perp}t}{2}\right) - ic_{\downarrow}(0)\sin\left(\frac{\Omega_{\perp}t}{2}\right)$$
$$c_{\downarrow}(t) = c_{\downarrow}(0)\cos\left(\frac{\Omega_{\perp}t}{2}\right) - ic_{\uparrow}(0)\sin\left(\frac{\Omega_{\perp}t}{2}\right)$$

Interpret this as an SU(2) rotation. What is the axis and angle of rotation?

(d) Now consider the case of general detuning. Make the Ansatz $c_{\uparrow}(t) = \tilde{c}_{\uparrow}(t)e^{-i\frac{\Delta}{2}t}$, $c_{\downarrow}(t) = \tilde{c}_{\downarrow}(t)e^{+i\frac{\Delta}{2}t}$. Show that the new amplitudes satisfy the differential equations

$$\dot{\tilde{c}}_{\uparrow}(t) = i\frac{\Delta}{2}\tilde{c}_{\uparrow}(t) - i\frac{\Omega_{\perp}}{2}\tilde{c}_{\downarrow}(t)$$
$$\dot{\tilde{c}}_{\downarrow}(t) = -i\frac{\Delta}{2}\tilde{c}_{\downarrow}(t) - i\frac{\Omega_{\perp}}{2}\tilde{c}_{\uparrow}(t)$$

(e) Find the solution to these equations off resonance. The answer you should find is

$$\tilde{c}_{\uparrow}(t) = \tilde{c}_{\uparrow}(0)\cos\left(\frac{\tilde{\Omega}t}{2}\right) + i\frac{\Delta\tilde{c}_{\uparrow}(0) - \Omega_{\perp}\tilde{c}_{\downarrow}(0)}{\tilde{\Omega}}\sin\left(\frac{\tilde{\Omega}t}{2}\right)$$
$$\tilde{c}_{\downarrow}(t) = \tilde{c}_{\downarrow}(0)\cos\left(\frac{\tilde{\Omega}t}{2}\right) - i\frac{\Delta\tilde{c}_{\downarrow}(0) + \Omega_{\perp}\tilde{c}_{\uparrow}(0)}{\tilde{\Omega}}\sin\left(\frac{\tilde{\Omega}t}{2}\right)$$

where $\tilde{\Omega} = \sqrt{\Omega_{\perp}^2 + \Delta^2}$

(f) (Extra Credit 5 points) Interpret this as an SU(2) rotation. What is the axis and angle of rotation?

(g) Suppose now we change the perturbing magnetic field to be solely oscillating along the $x$-direction, $\mathbf{B}_1 = B_{\perp}\cos\omega t\,\mathbf{e}_x$. Show that now the Hamiltonian is

$$\hat{H}(t) = \frac{\hbar\Omega_{\parallel}}{2}\hat{\sigma}_z + \frac{\hbar\Omega_{\perp}}{4}\left(\hat{\sigma}_+ e^{-i\omega t} + \hat{\sigma}_- e^{+i\omega t} + \hat{\sigma}_+ e^{+i\omega t} + \hat{\sigma}_- e^{-i\omega t}\right)$$

and the equations of motion in part (c) are now

$$\dot{c}_{\uparrow}(t) = -i\frac{\Omega_{\perp}}{4}c_{\downarrow}(t)e^{-i\Delta t} - i\frac{\Omega_{\perp}}{4}c_{\downarrow}(t)e^{+i(\omega+\Omega_{\parallel})t}$$
$$\dot{c}_{\downarrow}(t) = -i\frac{\Omega_{\perp}}{4}c_{\uparrow}(t)e^{+i\Delta t} - i\frac{\Omega_{\perp}}{4}c_{\uparrow}(t)e^{-i(\omega+\Omega_{\parallel})t}$$

Note: When $|\Delta|, \Omega_{\perp} << \omega, \Omega_{\parallel}$, the last terms in these differential equations oscillate much faster than any other scale in the problem. This rapid oscillations imply that, to good approximation, they average to zero and can be neglected. This is known as the "rotating wave approximation."