from interferences between different scattering amplitudes. The high intensity limit will be considered in Chapter 7, and the dipole force will be interpreted in this limit in terms of gradients of dressed state energies.

4. Moving atom . Friction force

We consider now an atom moving with a velocity $\mathbf{v}_0$, so that its position $\mathbf{r}$ is given by

$$\mathbf{r} = \mathbf{v}_0 t \tag{4.1}$$

if we take $\mathbf{r} = \mathbf{0}$ at $t = 0$.

4.1. Simple case of a laser plane wave

The laser wave is supposed to be a plane wave with wave vector $\mathbf{k}_L$ [see Equ. (3.20)]. Since the amplitude and the polarization of the laser electric field do not depend on $\mathbf{r}$, the Rabi frequency is position independent and consequently does not depend on time

$$\Omega_1(\mathbf{r} = \mathbf{v}_0 t) = \Omega_1 = \text{constant} \tag{4.2}$$

On the other hand, the phase Φ varies linearly with $\mathbf{r}$

$$\Phi(\mathbf{r}) = -\mathbf{k}_L \cdot \mathbf{r} \tag{4.3}$$

so that

$$\dot{\Phi} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \cdot \nabla\Phi = \mathbf{v}_0 \cdot \nabla\Phi = -\mathbf{k}_L \cdot \mathbf{v}_0 \tag{4.4}$$

Since Ω_1 and $\dot{\Phi}$ are time independent, optical Bloch equations are still a set of coupled linear differential equations with time independent coefficients. They thus have a steady-state solution which is derived from the solution obtained in section 3.1 (where $\mathbf{v}_0$ was equal to zero) by the substitution

$$\delta \longrightarrow \delta + \dot{\Phi} = \delta - \mathbf{k}_L \cdot \mathbf{v}_0 \tag{4.5}$$

or equivalently, since $\delta = \omega_L - \omega_A$, by the substitution

$$\omega_L \longrightarrow \omega_L - \mathbf{k}_L \cdot \mathbf{v}_0 \tag{4.6}$$

Such a result means that the atom moving with velocity $\mathbf{v}_0$ "sees " the laser frequency shifted by the *Doppler shift* $-\mathbf{k}_L \cdot \mathbf{v}_0$.

Figure 3.a represents an atom moving with velocity $\mathbf{v}_0$ in a laser plane wave propagating along the negative direction of the $0x$ axis, so that its wave vector can be written $\mathbf{k}_L = -k_L\,\epsilon_x$. If $v_0 = \epsilon_x \cdot \mathbf{v}_0$ is the projection of $\mathbf{v}_0$ along the $0x$ axis, we have $-\mathbf{k}_L \cdot \mathbf{v}_0 = k_L\,v_0$. Figure 3.b represents the component along $0x$ of the mean force experienced by the atom plotted versus $k_L v_0$ [we just replace in Equ. (3.25) δ by $\delta + k_L\,v_0$]. We have assumed that δ is negative. The force is negative and reaches its maximum value when $\delta = -k_L\,v_0$, i.e. when the apparent laser frequency $\omega_L + k_L\,v_0$ coincides with the atomic frequency ω_A. Near $v_0 = 0$ we can write

$$\mathcal{F}_x(v_0) = \mathcal{F}_x(v_0 = 0) - \alpha\,v_0 + \ldots \tag{4.7}$$

where the term linear in v_0 is a friction force , since it is proportional to v_0 with the opposite sign. The coefficient α is called the *friction coefficient*.

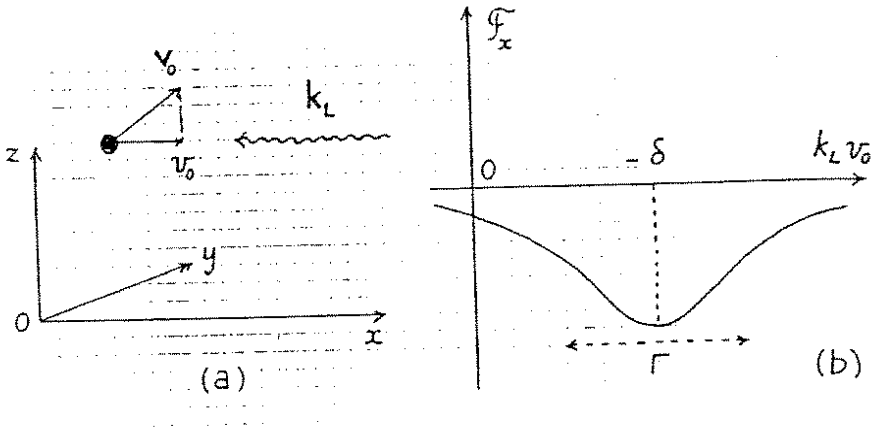

Fig. 3. (a) Atom moving with velocity $\mathbf{v}_0$ in a laser plane wave with wave vector $\mathbf{k}_L$. (b) Mean force experienced by the atom versus $k_L v_0$.

Replacing δ by $\delta - k_L\,v_0$ in (3.25) and expanding the result obtained for $\mathcal{F}_x(v_0)$ in powers of $k_L\,v_0/\Gamma$, we get for α the following expression

$$\alpha = -\hbar k_L^2\,\frac{s}{(1+s)^2}\,\frac{\delta\,\Gamma}{\delta^2 + (\Gamma^2/4)} \tag{4.8}$$

where s is the saturation parameter defined in (3.6). It clearly appears on (3.8) that α is positive for $\delta < 0$. For a fixed value of s, the value of δ

which optimizes (4.8) is $\delta = -\Gamma/2$. Taking this value of δ, we can then look for the optimal value of s which is found to be $s = 1$, corresponding to $\Omega_1 = \Gamma$. Taking these values of δ and Ω_1, we determine the maximum value of α

$$\alpha_{\max} = \frac{\hbar k_L^2}{4} \tag{4.9}$$

If we come back to Equ. (4.7), and if we suppose that $\mathcal{F}_x(v_0 = 0)$ is compensated for by a static external force, we find that the equation of motion of the center of the atomic wave packet is

$$M\frac{\mathrm{d}v_0}{\mathrm{d}t} = \mathcal{F} = -\alpha\, v_0 \tag{4.10}$$

which means that the atomic velocity is damped with a rate

$$\gamma = \frac{\alpha}{M} = \frac{\hbar k_L^2}{4M} = \frac{E_R}{2\hbar} \tag{4.11}$$

where E_R is the recoil energy defined in (1.8). We have thus established the result announced in section 1.3 according to which external variables, such as the atomic velocity, have a characteristic damping time on the order of $T_{\text{ext}} = \gamma^{-1} \simeq \hbar/E_R$.

4.2. Laser standing wave

The laser wave is supposed now to be a standing wave along the $0x$ axis, linearly polarized along $0z$, so that (1.2) becomes

$$\mathbf{E}_L(\mathbf{r}, t) = \epsilon_z\, \mathcal{E}_0(x) \cos \omega_L t \tag{4.12}$$

where the amplitude $\mathcal{E}_0(x)$ is given by

$$\mathcal{E}_0(x) = 2\, \mathcal{E}_0 \cos k_L x \tag{4.13}$$

Inserting (4.13) into (4.12), we get

$$\mathbf{E}_L(\mathbf{r}, t) = \epsilon_z\, \mathcal{E}_0[\cos(\omega_L t - k_L x) + \cos(\omega_L t + k_L x)] \tag{4.14}$$

which shows that the laser wave can be considered as the superposition of two counterpropagating plane waves, with the same amplitude $\mathcal{E}_0$. It must be emphasized however that the force exerted by the standing

wave is not simply the sum of the radiation pressures of the two counterpropagating waves. There are interference effects between these two waves which play an essential role.

In a standing wave, the phase of the field is the same everywhere, so that $\beta = \nabla\Phi = 0$. On the other hand, the Rabi frequency is position dependent and can be written

$$\Omega_1(x) = -\frac{d\,\mathcal{E}_0(x)}{\hbar} = 2\,\Omega_1 \cos k_L x \tag{4.15}$$

where

$$\Omega_1 = -\frac{d\,\mathcal{E}_0}{\hbar} \tag{4.16}$$

is the Rabi frequency associated with each of the two counterpropagating plane waves forming the standing wave. It follows that

$$\alpha = \frac{\nabla\Omega_1(x)}{\Omega_1(x)} = \frac{\nabla\mathcal{E}_0(x)}{\mathcal{E}_0(x)} = -2k_L \tan k_L x\ \epsilon_x \tag{4.17}$$

is different from zero. According to Equ. (2.37), the mean force experienced by the atom depends only on the component u of the Bloch vector.

In order to find u, we have to solve optical Bloch equations (2.46). If the atom is moving whith velocity v_0 along $0x$, we can replace x by $v_0 t$. We then see on Equ. (4.15) that $\Omega_1(x)$ becomes a sinusoidal function of t, with frequency $k_L v_0$. On the other hand, $\dot{\Phi}$ vanishes since Φ does not depend on x. We conclude that , for an atom moving in a standing wave, optical Bloch equations form a set of coupled linear differential equations with coefficients depending sinusoidally on time. Contrarily to what happens for a plane wave, it is in general impossible to solve analytically these equations, and we must use some approximations.

4.2.1. Limit of small velocities ($k_L v_0 \ll \Gamma$)

We present here a method of resolution of optical Bloch equations, first introduced in Ref. 8, and which consists of looking for an expansion of the solution in powers of kv_0/Γ. The zeroth order term represents the "adiabatic" solution, corresponding to a situation where the atom is moving so slowly along $0x$ that its internal state, when it passes in x, is the same as the one associated with an atom at rest in x. The first order term gives the first correction to the adiabatic approximation. It is linear in v_0, more precisely in kv_0/Γ which is the nonadiabaticity

parameter, equal to the ratio between the distance $v_0\Gamma^{-1}$ over which the atom travels during the internal response time Γ^{-1} and the laser wavelength k^{-1} which characterizes the spatial variations of the laser field. When inserted into the expression (2.37) of the force, this first order correction gives rise to a force linear in v_0, which is precisely the friction force.

In order to find the terms linear in v in the solution of optical Bloch equations , we first write Equations (2.46) in a compact matrix form

$$(\dot{X}) = (\frac{\partial}{\partial t} + v_0 \frac{\partial}{\partial x})(X) = (\mathcal{B})(X) - (X_s) \tag{4.18}$$

where the column vectors X (Bloch vector) and X_s (source term) and the square matrix $\mathcal{B}$ (Bloch matrix) are given by

$$(X) = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \quad (X_s) = \begin{pmatrix} 0 \\ 0 \\ -\Gamma/2 \end{pmatrix} \quad (\mathcal{B}) = \begin{pmatrix} -\Gamma/2 & \delta & 0 \\ -\delta & -\Gamma/2 & -\Omega_1(x) \\ 0 & \Omega_1(x) & -\Gamma \end{pmatrix} \tag{4.19}$$

In(4.18), we have used the "hydrodynamic derivative" $\mathrm{d}/\mathrm{d}t = (\partial/\partial t) + v_0(\partial/\partial x)$.After a transient regime which lasts a time on the order of Γ^{-1}, the contributions of $\partial X/\partial t$ vanishes and we have

$$v_0 \frac{\partial}{\partial x}(X) = (\mathcal{B})(X) - (X_s) \tag{4.20}$$

We now insert the expansion

$$(X) = (X^{(0)}) + (X^{(1)}) + \ldots \tag{4.21}$$

of X in powers of kv_0/Γ into (4.20). To order 0 in kv_0/Γ, the left-hand side vanishes and we get

$$0 = (\mathcal{B})(X^{(0)}) - (X_s) \tag{4.22}$$

or equivalently

$$(X^{(0)}) = (\mathcal{B})^{-1}(X_s) \tag{4.23}$$

which is just the steady-state Bloch vector for an atom at rest in x. To order 1 in kv_0/Γ, we then get

$$v_0 \frac{\partial}{\partial x}(X^{(0)}) = (\mathcal{B})(X^{(1)}) \tag{4.24}$$

which can be transformed, using (4.23) into

$$(X^{(1)}) = (\mathcal{B})^{-1} v_0 \frac{\partial}{\partial x}(X^{(0)}) = (\mathcal{B})^{-1} v_0 \frac{\partial}{\partial x}(\mathcal{B})^{-1}(X_s) \qquad (4.25)$$

Finally, we insert the expansion $u = u^{(0)} + u^{(1)} + \ldots$ of the first component of X into the expression (2.37) of the force . We don't give here the expression of the friction force which results from such a calculation, since it can be found in Ref. 8. We just point out a few important characteristics of such a friction force.

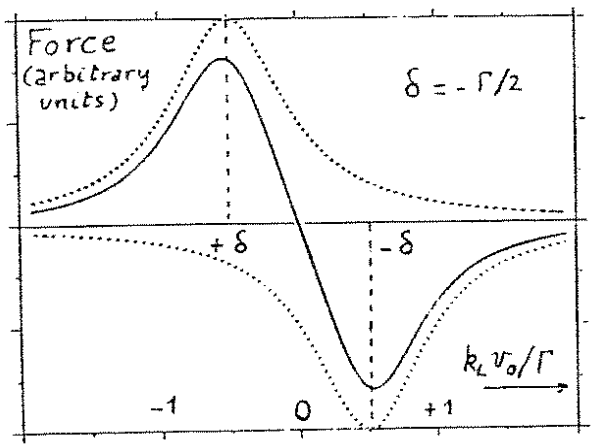

Fig. 4. Principe of Doppler cooling. Because of the Doppler effect, the radiation pressures exerted by the two counterpropagating waves forming the standing wave get unbalanced (curves in dotted lines), resulting in a net force opposite to the atomic velocity (curve in solid line).

Consider first the weak intensity limit: $s_0 \ll 1$, where s_0 is the saturation parameter associated with each of the two couterpropagating waves forming the standing wave. One finds in this limit that the friction force, averaged over one wavelength, coincides with the sum of the two friction forces exerted by the two counterpropagating plane waves. It thus appears that, at weak intensity, the interference effects between the two counterpropagating waves acting upon a moving atom vanish when averaged over one wavelength. Such an important result will be

rederived in Section 6.5 from a different point of view using scattering amplitudes. It actually provides the justification for the physical picture usually given for Doppler cooling[13]. Consider an atom moving with velocity v_0 along a weak standing wave, formed by two counterpropagating waves $+\mathbf{k}_L$ and $-\mathbf{k}_L$, slightly detuned to the red side of the atomic frequency (Fig. 4). Because of the Doppler effect, the atom gets closer to resonance with the wave opposing its motion, farther from resonance with the other wave, so that the two forces exerted by the two waves become unbalanced, resulting in a net force opposite to v_0 .

At high intensity ($s_0 \gg 1$), one finds that, contrarily to what happens at low intensity, the friction force $\mathcal{F}(x,v)$, averaged over one wavelength, becomes an "antidamping" force for a red detuning ($\omega_L < \omega_A$) and a friction force for a blue detuning ($\omega_L > \omega_A$). The physical interpretation of such a surprising result will be given later on, using the dressed-atom approach (see Chapter7, Section 7.4).

4.2.2. Arbitrary velocity. Method of continued fractions.

When kv becomes on the order of Γ, or larger than Γ, it is no longer possible to use the expansion (4.21) of the Bloch vector in powers of kv_0/Γ. We present in this section another approach to the problem, first introduced in Ref.14, and using the fact that the components of the Bloch vector are periodic functions of time, which itself is a consequence of the sumsoidal dependance on t of $\Omega_1(x) = \Omega_1(v_0 t)$ [see Equ.(4.15)].

Since u, v, w are periodic functions of time, we can expand them in Fourier series. Inserting these expansions in optical Bloch equations (4.18) leads to a set of recurrence relations between the various coefficients of the Fourier series expansions of u, v, w. It turns out that these equations can be formally solved in terms of continued fractions, which are very convenient for computer calculations. We will not give here the details of such calculations, which can be found in Ref. 14. We just mention a few important results obtained in this way, and give their physical meaning.

In Figure 5, we have sketched the variations with $k_L v_0$ of the static part of the force $\mathcal{F} = -\hbar\Omega_1(x)\,u(x,v)\,\boldsymbol{\alpha}$, averaged over one wavelength. We have supposed a red detuning ($\delta < 0$), and a intensity high enough so that the slope in $v_0 = 0$ is positive, contrarily to what happens at low intensity where we have a friction force for $\delta < 0$ (see end of the previous subsection 4.2.1). For larger values of $k_L v_0 v$, we see on Fig.5 that the force changes sign and exhibits resonant variations around values of $k_L v_0$ which, after extrapolation to zero laser intensity, correspond to $kv_0 = -\delta/3, -\delta/5...$

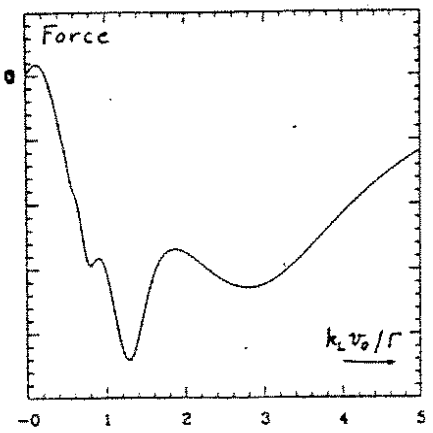

Fig. 5. Variations with $k_L v_0$ of the static part of the mean force experienced by an atom moving in a high intensity standing wave. This curve corresponds to $\delta = -2\Gamma$, $\Omega_1 = 5\Gamma$. The radiative shifts of the Doppleron resonances are already quite large.

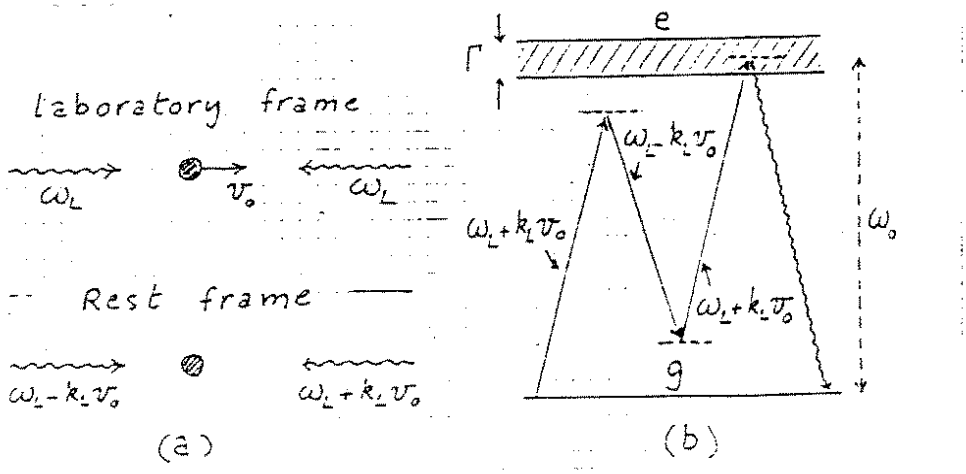

Fig. 6. (a) Frequencies of the two counterpropagating waves forming the standing wave in the laboratory frame and in the rest frame. (b) Resonant multiphoton process responsible for the resonance $k_L v_0 = -\delta/3$.

The resonances appearing on Fig. 5 can be simply interpreted in terms of resonant multiphoton processes. For example, Figure 6 represents such a multiphoton process, responsible for the resonance $k_L v_0 = -\delta/3$.

In the atomic rest frame, the apparent frequencies of the two counter-propagating waves forming the standing wave are Doppler shifted to $\omega_L + k_L v_0$ and $\omega_L - k_L v_0$ respectively (Fig.6.a). The atom can make a resonant transition from g to e by a three-photon process involving the absorption of one $\omega_L + k_L v_0$ photon, the stimulated emission of one $\omega_L - k_L v_0$ photon and the absorption of a second $\omega_L + k_L v_0$ photon (straight arrows of Fig.6.b). Such a process is resonant if

$$2(\omega_L + k_L v_0) - (\omega_L - k_L v_0) = \omega_A \tag{4.26}$$

i.e. if

$$\omega_L - \omega_A = -3k_L v_0 \tag{4.27}$$

The width of the resonance is determined by the natural width Γ of the upper state. Once in e, the atom falls back in g by a spontaneous emission process (wavy arrow of Fig.6.b). Similar diagrams involving $n+1$ absorptions, n stimulated emissions and one spontaneous emission could be given for explaining the resonance $k_L v_0 = -\delta/(2n+1)$.Such resonant multiphoton processes involving photons with Doppler shifted frequencies are called "Dopplerons"[15].

4.3. The σ^+- σ^- configuration for a $J_g = 0 \leftrightarrow J_e = 1$ transition.

Figure 6.b clearly shows that redistribution processes play an important role in the resonances appearing on Fig. 5, since we have absorption of photons from one wave followed by stimulated emission of photons in the counterpropagating wave. Actually, all the difficulties encountered in the theoretical description of the force experienced by an atom in a standing wave come from these redistribution processes which involve interference effects between the two waves. We will come back to these problems in Chapter 6. We present now another laser configuration and another atomic transition, for which there are no redistribution processes, and which consequently lead to much simpler results for the velocity dependent force[7].

We consider an atom with a transition $J_g = 0 \leftrightarrow J_e = 1$, having a single Zeeman sublevel g_0 in the ground state and three Zeeman sublevels e_{-1},e_0 and e_{+1} in the excited state (Fig.7.a). This atom is moving with velocity v along the same axis as two counterpropagating waves having respectively a right circular (σ^+) and a left circular (σ^-) polarization (Fig.7.b). Because of angular momentum conservation, the σ^+ wave excites only the transition $g_0 \leftrightarrow e_{+1}$ and the σ^- wave excites only the

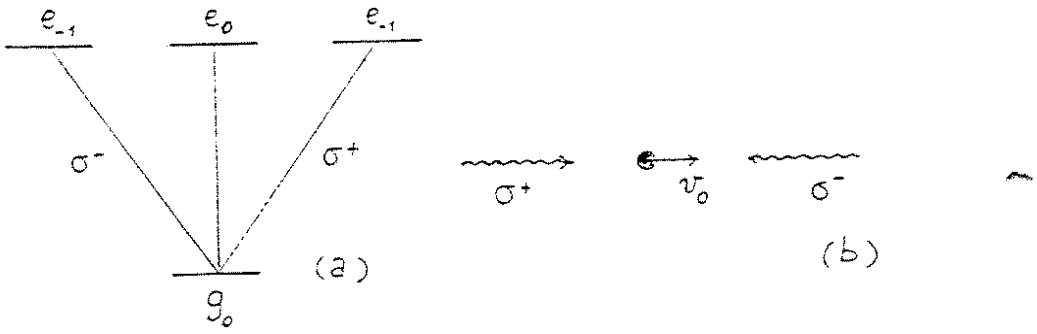

Fig. 7. (a) Zeeman sublevels for an atomic transition $J_g = 0 \longleftrightarrow J_e = 1$. (b) Laser configuration formed by two counterpropagating waves having respectively a right circular (σ^+) and a left circular (σ^-) polarization.

transition $g_0 \leftrightarrow e_{-1}$. It follows that if the atom absorbs a σ^+ photon and goes from g_0 to e_{+1}, it cannot come back to g_0 by stimulated emission of a photon in the other σ^- wave. In other words, conservation of angular momentum prevents any redistribution of photons between the two counterpropagating waves. This explains why the calculation of the velocity dependent force is, in this case, much simpler than for a two-level atom. As shown in Ref. 7, the force can be exactly calculated. For a red detuning ($\omega_L < \omega_A$), it remains a friction force for all intensities. No resonances, corresponding to Doppleron multiphoton processes, appear in the curve giving the variations of the force with the velocity.

5. Fluctuations of radiative forces

After having studied in the preceding two chapters 3 and 4 the mean value $\mathcal{F}$ of the force, we consider now the fluctuating part $\delta\mathbf{F}$ of this force given by (2.32) and (2.33). Such a fluctuating force introduces noise in atomic motion and is responsible for a diffusion of atomic momentum which limits the efficiency of laser cooling and laser trapping. In this chapter, we explain how it is possible to describe theoretically such fluctuations and we discuss the physical content of the results.

Since atomic motion in laser light presents great similarities with Brownian motion, we have thought it would be useful to recall first

(Section 5.1) a few basic results concerning classical Brownian motion. We then approach the problem of fluctuations of radiative forces, first in the Heisenberg picture (Section 5.2), where we follow closely the presentation of Ref. 8, then in the Schrödinger picture (Section 5.3), where we summarize the results derived in Ref. 17.

5.1. Classical Brownian motion[16]

5.1.1. Langevin equation

In order to describe the random motion of a heavy particle, with mass M and momentum $\mathbf{p}$, immersed in a fluid of light particles, Langevin introduced the following equation (for each component p of $\mathbf{p}$)

$$\frac{\mathrm{d}}{\mathrm{d}t}p(t) = -\gamma p(t) + F(t) \tag{5.1}$$

The total force acting on the particle is split into two parts : a friction force, $-\gamma p(t)$, representing the cumulative effect of collisions which damp the particle momentum with a "relaxation time"

$$T_R = \gamma^{-1} \tag{5.2}$$

and a fluctuating force $F(t)$, called the "Langevin force", responsible for the fluctuations of $p(t)$ about its mean value. In (5.1), $F(t)$ is considered as an external force, independent of $p(t)$, having a zero average value

$$\overline{F(t)} = 0 \tag{5.3.a}$$

and a correlation function equal to

$$\overline{F(t)F(t')} = 2D\, g(t-t') \tag{5.3.b}$$

where D is a coefficient which will be interpreted later on, and where $g(t-t')$ is a normalized function of $t-t'$

$$\int_{-\infty}^{+\infty} g(\tau)\, d\tau = 1 \tag{5.4}$$

which is an even function of $t-t'$ (since $F(t)$ is stationary), and which has a width on the order of the collision time τ_c.

Usually, the collision time τ_c is much smaller than the relaxation time T_R

$$\tau_c \ll T_R \tag{5.5}$$

which means that we have two well separated time scales in the problem. It follows that, with respect to functions of $t - t'$ varying with a characteristic time T_R, Eq.(5.3.b) can be approximated, taking into account (5.4), by

$$\overline{F(t)F(t')} \simeq 2D\,\delta(t - t') \tag{5.6}$$

5.1.2. Momentum diffusion coefficient

The solution of Eq.(5.1) corresponding to $p(t_0) = p_0$ can be written

$$p(t) = p_0 e^{-\gamma(t-t_0)} + \int_{t_0}^{t} \mathrm{d}t'\, F(t')\, e^{-\gamma(t-t')} \tag{5.7}$$

From (5.3.a), it follows that

$$\overline{p(t)} = p_0 e^{-\gamma(t-t_0)} \tag{5.8}$$

which means that the mean momentum of the particle is damped with a time constant $T_R = \gamma^{-1}$.

We now evaluate the variance $\sigma_p^2(t)$ of p

$$\sigma_p^2(t) = \overline{\left[p(t) - \overline{p(t)}\right]^2} \tag{5.9}$$

Using (5.7),(5.8), (5.3.b) and (5.6), and assuming $t - t_0 \gg \tau_c$, we get

$$\begin{aligned} \sigma_p^2(t) &= \int_{t_0}^{t} \mathrm{d}t' \int_{t_0}^{t} \mathrm{d}t''\, \overline{F(t')F(t'')}\, e^{-\gamma(t-t')} e^{-\gamma(t-t'')} \\ &\simeq \frac{D}{\gamma}\left[1 - e^{-2\gamma(t-t_0)}\right] \end{aligned} \tag{5.10}$$

For time intervals short compared to the relaxation time ($t - t_0 \ll \gamma^{-1}$), we can expand the exponential of (5.10) and we get

$$\tau_c \ll t - t_0 \ll \gamma^{-1} \qquad \longrightarrow \qquad \sigma_p^2(t) \simeq 2D\,(t - t_0) \tag{5.11}$$

It follows that, at short times, the variance of $p(t)$ increases linearly with $t - t_0$, with a rate $2D$. This shows that D is a momentum diffusion

coefficient. For long time intervals ($t - t_0 \gg \gamma^{-1}$), the exponential of (5.10) becomes negligible and we get, using (5.8)

$$t - t_\circ \gg \gamma^{-1} \qquad \longrightarrow \qquad \sigma_p^2 = \overline{p^2} - \overline{p}^2 = \overline{p^2} = \frac{D}{\gamma} \tag{5.12}$$

The variance of p tends to a fixed value equal to D/γ. On the other hand, if the particle is assumed to reach an equilibrium at the temperature T of the surrounding fluid, we have

$$\frac{\overline{p^2}}{2M} = \frac{1}{2} k_B T \tag{5.13}$$

Equating (5.12) and (5.13) gives the Einstein's equation

$$D = M\gamma k_B T \tag{5.13}$$

which relates the fluctuations of F, characterized by D, to the damping rate γ which characterizes the dissipative force damping the particle momentum.

5.1.3. Classical regression theorem

In this section, we present a simple method for calculating the correlation function of $p(t)$, which can be easily extended to quantum correlation functions.

We first calculate the correlation function $\overline{F(t)p(t')}$, involving both $F(t)$ and $p(t')$. If we let t_0 go to $-\infty$ in (5.7), we get a steady-state correlation function given by

$$\overline{F(t)\,p(t')} = \int_{-\infty}^{t'} \mathrm{d}t'' \; \underbrace{\overline{F(t)F(t'')}}_{2D\delta(t-t'')} e^{-\gamma(t'-t'')} \tag{5.15}$$

If $t \gg t'$, more precisely if $t - t' \gg \tau_c$, the delta function $\delta(t - t'')$ of (5.15) is outside the domain of integration, so that

$$t - t' \gg \tau_c \qquad \longrightarrow \qquad \overline{F(t)\,p(t')} = 0 \tag{5.16}$$

Such a result means that $p(t')$, which depends on the Langevin force $F(t'')$ in the past of t', cannot be correlated with the Langevin force $F(t)$ in the future of t'. On the other hand, if $t \ll t'$, more precisely if $t' - t \gg \tau_c$, we can extend to $+\infty$ the upper limit of the integral of (5.15) which is then easily evaluated

$$t' - t \gg \tau_c \qquad \longrightarrow \qquad \overline{F(t)\,p(t')} = 2D\, e^{-\gamma(t-t')} \tag{5.17}$$

Finally, for t close to t', $\overline{F(t)\,p(t')}$ varies rapidly between $2D$ and 0 over an interval of width τ_c, taking the value D for $t = t'$ (since $\overline{F(t)F(t'')}$ is an even function of $t - t''$). All these results are regrouped on Figure 8.

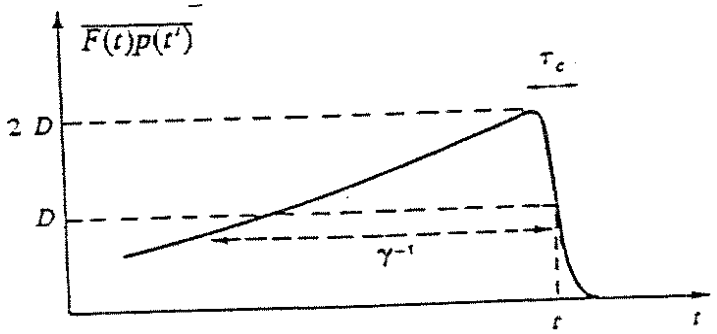

Fig. 8. Variations of $\overline{F(t)\,p(t')}$ versus t.

We calculate now the autocorrelation function $\overline{p(t)\,p(t')}$ of $p(t)$. For that, we multiply both sides of (5.1) by $p(t')$ and we take the average value

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\overline{p(t)\,p(t')} = -\gamma\,\overline{p(t)\,p(t')} + \overline{F(t)\,p(t')} \tag{5.18}$$

For $t-t' \gg \tau_c$, the last term of (5.18) vanishes, according to (5.16), and we get

$$t-t' \gg \tau_c \qquad \longrightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}t}\,\overline{p(t)\,p(t')} = -\gamma\,\overline{p(t)\,p(t')} \tag{5.19}$$

For $0 \leq t-t' \leq \tau_c$, $\overline{F(t)\,p(t')}$ is on the order of D, according to Fig. 8, so that the contribution of the last term of (5.18) to $\overline{p(t)\,p(t')}$, which is equal to the integral over t'' from t to t' of $\overline{F(t)\,p(t'')}$, remains bounded by $D\tau_c$. Since $\tau_c \ll \gamma^{-1}$, this contribution is very small compared to $D\gamma^{-1}$, which is nothing but the initial value $\overline{p^2}$ of $\overline{p(t)\,p(t')}$, for $t = t'$ (see Equ. (5.12)). We can therefore ignore, for $t > t'$, the last term of (5.18) and consider that the "two-time average " $\overline{p(t)\,p(t')}$ obeys the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\overline{p(t)\,p(t')} \simeq -\gamma\,\overline{p(t)\,p(t')} \tag{5.20}$$

which is quite similar to the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\overline{p(t)} = -\gamma\,\overline{p(t)} \tag{5.21}$$

obeyed by the "one-time average" $\overline{p(t)}$. In other words, the fluctuations "regress" as the mean values. Such a result can be extended to quantum correlation fonctions and is known as the "quantum regression theorem" (see Ref. 18 and Ref. 2, Complements C_{IV} and A_V).

5.1.4. Kramers-Fokker-Planck equation

Consider a Brownian particle moving in a one-dimensional potential well $U(x)$. Its position $x(t)$ and its momentum $p(t)$ obey the following equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = +\frac{p(t)}{M} \tag{5.22.a}$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\gamma p(t) - \frac{\mathrm{d}}{\mathrm{d}x}U(x) + F(t) \tag{5.22.b}$$

which are a straightforward extension of the Langevin equation (5.1). We want here to give the principle of the derivation of the evolution equation of the distribution function $\mathcal{P}(x,p,t)$, which is the probability density of finding the particle with position x and momentum p at time t.

It is clear on (5.22) that the rates of variation of $x(t)$ and $p(t)$ depend only on the state of the system at the same instant t, and not on its "history" in the past of t. For that reason, the stochastic process $\{x(t), p(t)\}$ is called a "Markov" process. Note that $\{p(t)\}$ alone would not be a Markov process, since $\mathrm{d}p(t)/\mathrm{d}t$ depends, according to (5.22.b) on $\mathrm{d}U(x)/\mathrm{d}x$, i.e. on $x(t) = \int_{\infty}^{t} \mathrm{d}t' p(t')/M$ which involves the whole past history of $p(t)$.

The distribution function $\mathcal{P}(x,p,t)$ associated with the Markov process $\{x(t), p(t)\}$ obeys the equation

$$\mathcal{P}(x,p,t) = \int\int \mathrm{d}x\,\mathrm{d}p\ \Pi(x,p,t/x',p',t')\,\mathcal{P}(x',p',t') \tag{5.23}$$

where $\Pi(x,p,t/x',p',t')$ is the conditional probability for ending in x,p at time t if one starts from x',p' at time t'. This probability is normalized, so that

$$\int\int \mathrm{d}x\,\mathrm{d}p\ \Pi(x,p,t/x',p',t') = 1 \tag{5.24}$$

We now choose a time interval δt such that $\tau_c \ll \delta t \ll T_R$. Replacing in (5.23) and (5.24) t by $t+\delta t$ and t' by t, we derive from (5.23) the

following equation

$$\begin{aligned}&\mathcal{P}(x,p,t+\delta t)-\mathcal{P}(x,p,t)=\\&=+\int\int \mathrm{d}x'\,\mathrm{d}p'\,\Pi(x,p,t+\delta t/x',p',t)\,\mathcal{P}(x',p',t)\\&\quad-\int\int \mathrm{d}x''\,\mathrm{d}p''\,\Pi(x'',p'',t+\delta t/x,p,t)\,\mathcal{P}(x,p,t)\end{aligned}\tag{5.25}$$

which has a clear physical meaning. After division by δt, such an equation expresses that the rate of variation of $\mathcal{P}(x,p,t)$ in x,p is equal to the rate in minus the rate out. It can thus be considered as a master equation for $\mathcal{P}(x,p,t)$.

We suppose now that the variations

$$\begin{aligned}\delta x&=x(t+\delta t)-x(t)\\\delta p&=p(t+\delta t)-p(t)\end{aligned}\tag{5.26}$$

of $x(t)$ and $p(t)$ between t and $t+\delta t$ are small compared to the widths Δx and Δp of $\mathcal{P}(x,p,t)$ in x and p respectively (limit of small jumps). By expanding in (5.25) $\mathcal{P}(x',p',t)$ in powers of $x'-x$ and $p'-p$, it is then possible to approximate the master equation (5.25) by a partial differential equation. If the Taylor series expansion of $\mathcal{P}(x',p',t)$ is stopped after order 2, one gets

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{P}(x_1,x_2,t)=&-\frac{\partial}{\partial x_1}M_1\mathcal{P}-\frac{\partial}{\partial x_2}M_1\mathcal{P}\\&+\sum_{i,j=1,2}\frac{\partial^2}{\partial x_i\partial x_j}D_{ij}\mathcal{P}\end{aligned}\tag{5.27}$$

where we have used the simplified notation $x_1=x, x_2=p$ and where M_i and D_{ij} are given by

$$M_i=\lim_{\tau_c\ll\delta t\ll T_R}\frac{\overline{\delta x_i}}{\delta t}\tag{5.28.a}$$

$$D_{ij}=\frac{1}{2!}\lim_{\tau_c\ll\delta t\ll T_R}\frac{\overline{\delta x_i\delta x_j}}{\delta t}\tag{5.28.b}$$

In order to calcule M_i and D_{ij} we come back to equations (5.22), and we derive, after calculations similar to those of subsection 5.1.2., the

following relations

$$\begin{aligned} \overline{\frac{\delta x}{\delta t}} &= \frac{p}{M} & \overline{\frac{\delta p}{\delta t}} &= -\gamma p - \frac{\mathrm{d}U}{\mathrm{d}x} \\ \overline{\frac{\delta x^2}{\delta t}} &= 0 = \overline{\frac{\delta x \delta p}{\delta t}} & \overline{\frac{\delta p^2}{\delta t}} &= 2D \end{aligned} \tag{5.29}$$

which inserted into (5.27) give the Kramers-Fokker-Planck equation

$$\begin{aligned} \frac{\partial}{\partial t}\mathcal{P}(x,p,t) = &- \frac{p}{M}\frac{\partial}{\partial x}\mathcal{P}(x,p,t) \\ &+ \frac{\partial}{\partial p}\left[\gamma p + \frac{\mathrm{d}U}{\mathrm{d}x}\right]\mathcal{P}(x,p,t) + D\frac{\partial^2}{\partial p^2}\mathcal{P}(x,p,t) \end{aligned} \tag{5.30}$$

5.2. Analysis of momentum diffusion in the Heisenberg picture

5.2.1. Momentum diffusion coefficient and Langevin force operator

By definition, the momentum diffusion coefficient D is related to the rate of increase of the momentum variance

$$2D = \frac{\mathrm{d}}{\mathrm{d}t}\Delta P^2(t) \tag{5.31}$$

with

$$\Delta P^2(t) = \left\langle \left[\mathbf{P}(t) - \langle \mathbf{P}(t)\rangle\right]^2 \right\rangle \tag{5.32}$$

Starting from the Heisenberg equation for $\mathbf{P}(t)$, given in Section 2 ,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{P}(t) = \mathbf{F}(t) \tag{5.33}$$

where $\mathbf{F}(t)$ is the force operator, we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left\langle \left[\mathbf{P}(t) - \langle \mathbf{P}(t)\rangle\right]^2 \right\rangle &= \frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{P}^2\rangle - 2\langle \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{P}\rangle \cdot \langle \mathbf{P}\rangle \\ &= \left\langle \mathbf{F}\cdot\mathbf{P} + \mathbf{P}\cdot\mathbf{F} \right\rangle - 2\left\langle \mathbf{F}\right\rangle \cdot \left\langle \mathbf{P}\right\rangle \end{aligned} \tag{5.34}$$

Inserting into (5.34) the solution

$$\mathbf{P}(t) = \int_0^\infty \mathrm{d}\tau\ \mathbf{F}(t-\tau) \tag{5.35}$$

of (5.33), and using (5.31) leads to

$$\begin{aligned} 2D &= 2\ \mathrm{Re} \int_0^\infty \mathrm{d}\tau\ \left[\langle \mathbf{F}(t) \cdot \mathbf{F}(t-\tau) \rangle - \langle \mathbf{F}(t) \rangle \cdot \langle \mathbf{F}(t-\tau) \rangle \right] \\ &= 2\ \mathrm{Re} \int_0^\infty \mathrm{d}\tau\ \langle\, \delta\mathbf{F}(t) \cdot \delta\mathbf{F}(t-\tau) \,\rangle \end{aligned} \tag{5.36}$$

where $\delta\mathbf{F}(t)$ is the fluctuating part of $\mathbf{F}(t)$

$$\delta\mathbf{F}(t) = \mathbf{F}(t) - \langle \mathbf{F}(t) \rangle \tag{5.37}$$

It thus appears that the momentum diffusion coefficient is related to the time integral of the correlation function of the Langevin force operator.

In the following subsection, we give a brief outline of the calculation of such a correlation function, leading to the expression (5.44) for D. The physical interpretation of this expression is then given in Subsection 5.2.3. The reader, not interested in the method of calculation of (5.36), can thus directly proceed to Subsection 5.2.3.

5.2.2. Correlation function of the Langevin force operator

We have seen in Chapter 2 (see (2.32) and (2.33)) that the Langevin force $\delta\mathbf{F}(\mathbf{r},t)$ is the sum of two forces $\delta\mathbf{F}_{\mathrm{las}}(\mathbf{r},t)$ and $\delta\mathbf{F}_{\mathrm{vac}}(\mathbf{r},t)$ representing respectively the contributions of the laser field and the vacuum field to the Langevin force. From the expression (2.4) of V_{AL} and the fact that only the z-component of $\mathbf{d}$ is non-zero, we easily get

$$\delta\mathbf{F}_{\mathrm{las}} = [\, d_z - \langle d_z \rangle] \nabla E_{Lz} \tag{5.38}$$

where $\delta d_z = d_z - \langle d_z \rangle$ is the fluctuating part of d_z. A similar calculation starting from (2.5) and (2.14), allows one to transform (2.33.b) into

$$\delta\mathbf{F}_{\mathrm{vac}} = d_z \nabla (E_z^{\mathrm{vac}})^+ + \left[\nabla (E_z^{\mathrm{vac}})^- \right] d_z \tag{5.39}$$

Replacing $\delta\mathbf{F}$ by $\delta\mathbf{F}_{\mathrm{las}} + \delta\mathbf{F}_{\mathrm{vac}}$ in (5.36) shows that the correlation function $\langle\, \delta\mathbf{F}(t) \cdot \delta\mathbf{F}(t-\tau) \,\rangle$ is a sum of 3 contributions, one involving only $\delta\mathbf{F}_{\mathrm{las}}$, one involving only $\delta\mathbf{F}_{\mathrm{vac}}$ and one involving both $\delta\mathbf{F}_{\mathrm{las}}$ and $\delta\mathbf{F}_{\mathrm{vac}}$. One can show that this last crossed term is zero as a result of the fact that the vacuum free field gradients, appearing in (5.39) and evaluated at time t, commute with the dipole operator $d_z(t')$ evaluated

at any other time t' (*). This allows one to put $\nabla(E_z^{\rm vac})^+$ at the extreme right, and $\nabla(E_z^{\rm vac})^-$ at the extreme left, which then gives zero when the vacuum average value is taken (see (2.15)). This shows that D can be written

$$D = D_{\rm las} + D_{\rm vac} \tag{5.40}$$

where

$$D_{\rm las} = 2\ {\rm Re} \int_0^\infty {\rm d}\tau\ \langle\, \delta \mathbf{F}_{\rm las}(t) \cdot \delta \mathbf{F}_{\rm las}(t-\tau) \,\rangle \tag{5.41}$$

$$D_{\rm vac} = 2\ {\rm Re} \int_0^\infty {\rm d}\tau\ \langle\, \delta \mathbf{F}_{\rm vac}(t) \cdot \delta \mathbf{F}_{\rm vac}(t-\tau) \,\rangle \tag{5.42}$$

In (5.38), ∇E_{Lz} is a c-number, so that the correlation function of $\delta \mathbf{F}_{\rm las}$ is proportional to the correlation function of δd_z. To calculate $\langle \delta d_z(t) \delta d_z(t-\tau) \rangle$, one can first express δd_z in terms of $\delta \Pi_{eg}$ and $\delta \Pi_{ge}$ where $\Pi_{ab} = \mid a \rangle\langle b \mid$ with $a, b = e$ *or* g(see (2.39)). Now, we have already mentioned in Section 2.5 that the equation of motion of Π_{ab} has the structure of a Langevin equation with damping terms and a Langevin force (see (2.41)). Multipling both sides of the equation giving $\dot{\Pi}_{ab}(t)$ by $\Pi_{cd}(t')$, and taking the vacuum average value, one can then show, by an argument very similar to the one used above in Subsection 5.1.3, that, for $t > t'$, the two-time averages $\langle \Pi_{ab}(t) \Pi_{cd}(t') \rangle$ obey the same equations as the one-time averages $\langle \Pi_{ab}(t) \rangle$ (c and d being fixed), i.e. optical Bloch equations. Such an important result, known as the "quantum regression theorem" (see Ref. 18 and Ref. 2, Complements C_{IV} and A_V) means that the correlation functions of the atomic dipole moment can be calculated from optical Bloch equations.

It remains to evaluate (5.42). Inserting (5.39) into (5.42), one sees that, due to (2.15), the only non-zero term is

$$\langle\, 0 \mid d_z(t) \left[\nabla(E_z^{\rm vac})^+(t)\right] \left[\nabla(E_z^{\rm vac})^-(t-\tau)\right] d_z(t-\tau) \mid 0 \,\rangle \tag{5.43}$$

The order of the two field operators appearing in (5.43) can be changed using their commutator, which is a c-number. The contribution of this commutator is therefore proportional to the correlation function of d_z,

(*) To demonstrate such a result, one can express the annihihation operators $a_j(t)$ appearing in the field gradients as a function of $a_j(t')$ and of the source field radiated between t' and t. $a_j(t')$ commutes with $d_z(t')$. The remaining commutator appears as an integral over the modes $\mathbf{k}\epsilon$ of an odd function of $\mathbf{k}$ and thus vanishes.

which can be calculated from optical Bloch equations. The remaining term can be transformed, using the fact mentioned above that the vacuum free field gradient operators commute with the dipole operators at any time. One can then easily show that this term is equal to zero, as a consequence of (2.15).

To summarize, we have shown that D is a sum of 2 terms. The first one, D_{las} , involves $\delta\mathbf{F}_{\text{las}}$ and thus $\boldsymbol{\nabla} E_{Lz}$, i.e. $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ given in (2.36). The second one, D_{vac} , comes from the commutator of the vacuum free field, i.e. from the quantum nature of this field, and is independent of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Both D_{las} and D_{vac} involve correlation functions of the atomic dipole moment which can be calculated from optical Bloch equations, using the quantum regression theorem.

5.2.3. Physical discussion

For a 2-level atom at rest in $\mathbf{r} = \mathbf{0}$, the method of calculation outlined in the previous subsection leads to the following result

$$\begin{aligned} D = &+ \hbar^2 k_L^2 \frac{\Gamma}{4} \frac{s}{1+s} \\ &+ \hbar^2 \boldsymbol{\beta}^2 \frac{\Gamma}{4} \frac{s}{(1+s)^3} \left\{ 1 + \frac{12\delta^2 - \Gamma^2}{4\delta^2 + \Gamma^2} s + s^2 \right\} \\ &+ \hbar^2 \boldsymbol{\alpha}^2 \frac{\Gamma}{4} \frac{s}{(1+s)^3} \left\{ 1 + \frac{-4\delta^2 + 3\Gamma^2}{4\delta^2 + \Gamma^2} s + 3s^2 + \frac{4\delta^2 + \Gamma^2}{\Gamma^2} s^3 \right\} \\ &- \hbar^2 \boldsymbol{\alpha} \cdot \boldsymbol{\beta}\ \delta \frac{s^2}{(1+s)^3} \left\{ \frac{4\Gamma^2}{4\delta^2 + \Gamma^2} + s \right\} \end{aligned} \tag{5.44}$$

where s is the saturation parameter in $\mathbf{0}$, given by (3.6), and where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are the logarithmic Rabi frequency gradient and phase gradient in $\mathbf{0}$ given in (3.3.b) and (3.3.c).

The first line of (5.44), which is independent of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is D_{vac}, which comes from the non commutation of the vacuum free field operators

$$D_{\text{vac}} = \hbar^2 k_L^2 \frac{\Gamma}{4} \frac{s}{1+s} \tag{5.45}$$

Such a term describes the momentum diffusion due to the random direction of the spontaneously emitted photons. The atomic momentum $\mathbf{P}$ accomplishes a random walk in momentum space, the size of each step being $\hbar k_L$, and the number of steps during δt being equal to $\Gamma \sigma_{ee}^{st} \delta t$, where σ_{ee}^{st} is the steady-state population of the upper state, given by

(3.7). It follows from the well-known properties of random walk that the increase of the variance of $\mathbf{p}$ during δt can be written

$$\begin{aligned} \overline{(\delta \mathbf{p}^2)} - \left(\overline{\delta \mathbf{p}}\right)^2 &= \hbar^2 k_L^2 \, \Gamma \, \sigma_{ee}^{st} \delta t \\ &= \hbar^2 k_L^2 \, \frac{\Gamma}{4} \, \frac{s}{1+s} \delta t \end{aligned} \tag{5.46}$$

The comparison of (5.45) and (5.46) then shows that the right-hand side of (5.46) can be written $2D_{\text{vac}}\delta t$, which confirms the physical interpretation of D_{vac}.

The three other lines of (5.44) correspond to D_{las}. For a laser plane wave, $\boldsymbol{\beta} = -\mathbf{k}_L$ and $\boldsymbol{\alpha} = \mathbf{0}$, so that only the second line of (5.44) contributes to D_{las}. Such a term then describes the momentum diffusion due to the fluctuations of radiation pressure, more precisely due to the fluctuations in the number of absorbed photons. It can be written, using (3.7), as

$$D_{\text{abs}} = \frac{1}{2} \hbar^2 k_L^2 \, \Gamma \, \frac{s}{1+s} (1+Q) \tag{5.47}$$

where Q is a dimensionless factor given by

$$Q = \frac{2\Omega_1^2 (4\delta^2 - 3\Gamma^2)}{(2\omega_1^2 + 4\delta^2 + \Gamma^2)^2} \tag{5.48}$$

In order to interpret (5.47), we introduce the number δN of laser photons absorbed during δt, and the corresponding momentum transferred to the atom $\delta \mathbf{p} = \hbar \mathbf{k}_L \delta N$ Since δN is a random variable,there is an increase of the atomic momentum variance due to the fluctuations in the number of absorbed photons

$$\overline{(\delta \mathbf{p}^2)} - \left(\overline{\delta \mathbf{p}}\right)^2 = \hbar^2 k_L^2 \left[\overline{\delta N^2} - \left(\overline{\delta N}\right)^2 \right] \tag{5.49}$$

Consider first the mean number of photons absorbed during δt, which is given by (see Equ. (3.17))

$$\overline{\delta N} = \Gamma \, \sigma_{ee}^{st} \, \delta t = \frac{\Gamma}{2} \, \frac{s}{1+s} \, \delta t \tag{5.50}$$

If δN was following a Poisson law, the variance of δN would be equal to $\overline{\delta N}$. Actually, this is not the case, and one can show[19] that there

are corrections to Poisson statistics in resonance fluorescence which are precisely described by the factor Q given in (5.48)

$$\overline{\delta N^2} - \left(\overline{\delta N}\right)^2 = \overline{\delta N}\,(1+Q) \tag{5.51}$$

It thus appears that (5.49) can be rewritten, using (5.51), (5.50) and (5.47) as

$$\overline{(\delta \mathbf{p}^2)} - \left(\overline{\delta \mathbf{p}}\right)^2 = 2\,D_{\mathrm{abs}}\,\delta t \tag{5.52}$$

which confirms the physical interpretation of D_{abs} as being due to the fluctuations in the number of absorbed photons. The comparison of (5.45) and (5.47) shows that D_{vac} and D_{abs} have the same order of magnitude.

In a laser standing wave, $\boldsymbol{\beta} = \mathbf{0}$, so that only the third line of (5.44) contributes to D_{las}. Such a term then describes the fluctuations of dipole forces and will be denoted by D_{dip}. We consider here a laser standing wave along the Ox axis, linearly polarized along Oz, and having a node in $x = 0$

$$\mathrm{E}_L(x,t) = \epsilon_z\, 2\mathcal{E}_0 \sin k_L x\, \cos \omega_L t \tag{5.53}$$

In (5.53), $\mathcal{E}_0$ is the amplitude of each of the two counterpropagating waves forming the standing wave, the corresponding Rabi frequency being equal to Ω_1. The standing wave Rabi frequency in x is thus equal to

$$\Omega_1(x) = 2\Omega_1 \sin k_L x \tag{5.54}$$

so that

$$\boldsymbol{\alpha} = \frac{\boldsymbol{\nabla}\Omega_1(x)}{\Omega_1(x)} = \epsilon_x\, k_L\, \frac{1}{\tan k_L x} \tag{5.55}$$

Finally, the saturation parameter in x, $s(x)$, is equal to

$$s(x) = 4 s_0 \sin^2 k_L x \tag{5.56}$$

where

$$s_0 = \frac{\Omega_1^2/2}{\delta^2 + (\Gamma^2/4)} = \frac{s_{\max}}{4} \tag{5.57}$$

is the saturation parameter of each of the two counterpropagating waves and $s_{\max}$ the maximum value of $s(x)$. We will not discuss here the general expression of D_{dip} obtained when (5.55) and (5.56) are inserted into the third line of (5.44). We will restrict ourselves to the low intensity limit ($s_0 \ll 1$) and to the high intensity limit ($s_0 \gg 1$).

If $s_0 \ll 1$, the third line of (5.44) reduces to

$$D_{\text{dip}} \simeq \hbar^2 k_L^2 \, \Gamma \, s_0 \cos^2 k_L x = \hbar^2 k_L^2 \, \Gamma \, \frac{s_{\max}}{4} \cos^2 k_L x \tag{5.58}$$

whereas the first line is equal to

$$D_{\text{vac}} \simeq \hbar^2 k_L^2 \, \Gamma \, s_0 \sin^2 k_L x \tag{5.59}$$

Near a node, for example near $x = 0$, we get a very surprising result. Since there is no light in a node, we expect that there are no fluorescence photons. Effectively, $D_{\text{vac}} \to 0$ if $x \to 0$. But D_{dip} takes in $x = 0$ its maximum value, equal to $\hbar^2 k_L^2 \, \Gamma \, s_0 = \hbar^2 k_L^2 \, \Gamma \, s_{\max}/4$. We will come back to this problem in Section 6.3 and show that, near a node, the large value of D_{dip} is due to a new kind of correlated redistribution.

If $s_0 \gg 1$, the third line of (5.44) tends to(*)

$$D_{\text{dip}} \simeq \hbar^2 k_L^2 \, \cos^2 k_L x \, \frac{\Omega_1^2}{2\Gamma} \tag{5.60}$$

Contrarily to D_{vac} and to D_{abs} given in (5.45) and (5.47), D_{dip} does not saturate at high laser intensities. A dressed atom interpretation of this result will be given in Section 7.3. We have seen above, in Section 3.4, that the depth of the optical potential well associated with dipole forces increases linearly with Ω_1 . The fact that the heating due to the fluctuations of dipole forces increases quadratically with Ω_1, as shown by (5.60), introduces severe limitations for laser traps. Note however that it is always possible, as suggested in Ref. 20, to alternate in time cooling and trapping phases.

5.2.4. The Doppler limit in laser cooling

The equilibrium temperature reached in laser cooling results from a competition between laser cooling which damps the atomic velocity with a rate $\gamma = \alpha/M$, where α is the friction coefficient (see (4.8))

$$\delta p/\delta t = -\gamma p \tag{5.61}$$

and the heating due to momentum diffusion

$$\left(\frac{\delta p^2}{\delta t}\right)_{\text{diffusion}} = 2D \tag{5.62}$$

(*) Equation (5.60) is not valid near a node, since $s(x) \to 0$ in such a place

From (5.61), it follows that

$$\left(\frac{\delta p^2}{\delta t}\right)_{\text{cooling}} = -2\gamma p^2 \tag{5.63}$$

In steady-state, the two rates (5.62) and (5.63) cancel out so that $\gamma p^2 \sim D$. The order of magnitude of the equilibrium temperature is thus

$$\frac{p^2}{2M} \sim k_B T \sim \frac{D}{M\gamma} \tag{5.64}$$

From (4.8), it follows that, at low intensity, and for $\delta = -\Gamma/2$, $\alpha \sim \hbar^2 k_L^2 \, s_0$, so that

$$\gamma \sim \frac{\hbar k_L^2}{M} s_0 \tag{5.65}$$

On the other hand, from the results derived in this section (see (5.45), (5.47) and (5.58)), we have, at low intensity

$$D \sim \hbar^2 k_L^2 \, \Gamma \, s_0 \tag{5.66}$$

Inserting (5.65) and (5.66) into (5.64) leads to

$$k_B T_D \sim \frac{D}{M\gamma} \sim \hbar\Gamma \tag{5.67}$$

It thus appears that the temperature reached by laser cooling for 2-level atoms is determined by Γ. The exact value of the minimum temperature which can be reached is given by $k_B T_D \sim \hbar\Gamma/2$ (see Refs. 8, 21, 22) and is called the Doppler limit. T_D is on the order of 240 μK for Na and 125 μK for Cs.

5.9. Quantum kinetic equation for the atomic Wigner function

We try now, using the Schrödinger picture, to derive an equation of motion for the reduced atomic density operator describing the translational degrees of freedom of the atom. We just give the outline of the derivation, putting the emphasis on the new results and on the new physical insights. More details may be found in Refs.17 and 23.

5.3.1. Atomic Wigner function

When both internal and external degrees of freedom are treated quantum-mechanically, the atomic density matrix is labelled by two types of quantum numbers. For example, if one uses the position representation for the center of mass, the atomic density matrix elements are $\langle i, \mathbf{r}'|\sigma|j, \mathbf{r}''\rangle$, where $i, j = e$ or g and $\mathbf{r}'$and $\mathbf{r}''$ are eigenvalues of the position operator $\mathbf{R}$. Similarly, if we use the momentum representation, we get $\langle i, \mathbf{p}'|\sigma|j, \mathbf{p}''\rangle$.

A very useful representation, which treats in a symmetrical way position and momentum, is the so called "Wigner representation" [24], which associates with the atomic density operator σ, a function of $\mathbf{r}$ and $\mathbf{p}$ given by

$$\begin{aligned} W_{ij}(\mathbf{r},\mathbf{p}) &= \frac{1}{h^3}\int \mathrm{d}^3u \exp(-i\mathbf{p}.\mathbf{u}/\hbar)\ \langle i, \mathbf{r}+\frac{\mathbf{u}}{2} \mid \sigma \mid j, \mathbf{r}-\frac{\mathbf{u}}{2}\rangle \\ &= \frac{1}{h^3}\int \mathrm{d}^3v \exp(+i\mathbf{r}.\mathbf{v}/\hbar)\ \langle i, \mathbf{p}+\frac{\mathbf{v}}{2} \mid \sigma \mid j, \mathbf{p}-\frac{\mathbf{v}}{2}\rangle \end{aligned} \tag{5.68}$$

From (5.68), one can introduce the Wigner function

$$f(\mathbf{r},\mathbf{p}) = W_{ee}(\mathbf{r},\mathbf{p}) + W_{gg}(\mathbf{r},\mathbf{p}) \tag{5.69}$$

which is a density of "quasi-probability" to find the atom in $\mathbf{r}$ with momentum $\mathbf{p}$, regardless of its internal state. The atomic Wigner function is real, normalized and appears as an ordinary probability density for all completely symmetrical functions of $\mathbf{R}$ and $\mathbf{P}$. For example,

$$\frac{1}{2}\langle\, \mathbf{R}.\mathbf{P} + \mathbf{P}.\mathbf{R}\,\rangle = \int \mathrm{d}^3r\, \mathrm{d}^3p\ \mathbf{r}.\mathbf{p}\ f(\mathbf{r},\mathbf{p}) \tag{5.70}$$

Note however that $f(\mathbf{r},\mathbf{p})$ can take negative values, which shows that it is not a true probability.

5.3.2. Generalized optical Bloch equations

These equations of motion generalize those discussed in Section 2.5 above, to the case where the external degrees of freedom are treated quantum-mechanically. They have the same structure as in 2.42, i.e. the sum of hamiltonian terms coming from the atomic hamiltonian H_A and the atom-laser interaction hamiltonian V_{AL}, and damping terms due to spontaneous emission. The new features are the appearance of the external quantum numbers.

For example, the equation of motion of $W_{gg}(\mathbf{r},\mathbf{p})$ is found to be

$$\frac{\partial}{\partial t}W_{gg}(\mathbf{r},\mathbf{p}) = -\frac{\mathbf{p}}{M}\cdot\frac{\partial}{\partial \mathbf{r}}W_{gg}(\mathbf{r},\mathbf{p}) + \frac{id}{\hbar}\int \mathrm{d}^3k\left[\mathcal{E}^-(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{r}}W_{eg}(\mathbf{r},\mathbf{p}-\frac{\hbar\mathbf{k}}{2}) - \mathcal{E}^+(\mathbf{k})e^{+i\mathbf{k}\cdot\mathbf{r}}W_{ge}(\mathbf{r},\mathbf{p}+\frac{\hbar\mathbf{k}}{2})\right] + \Gamma\int \mathrm{d}^2\kappa\,\phi(\kappa)\,W_{ee}(\mathbf{r},\mathbf{p}+\hbar k_A\kappa) \tag{5.71}$$

The first term comes from the commutator of σ with $H_A^{\mathrm{ext}} = \mathbf{P}^2/2M$, the second one from the commutator of σ with V_{AL}, $\mathcal{E}_L^+(\mathbf{k})$ and $\mathcal{E}_L^-(\mathbf{k})$ being respectively the Fourier transforms of the positive and negative frequency components of the laser field. Finally, the last term describes the feeding of the ground state from the excited state by spontaneous emission, $\phi(\kappa)$ being the relative probability of spontaneous emission of a photon in the direction κ and k_A being equal to ω_A/c.

It clearly appears on (5.71) that the atomic momentum undergoes discrete changes during the absorption and emission processes. It follows that the generalized optical Bloch equations are finite difference equations coupling the four functions $W_{ij}(\mathbf{r},\mathbf{p})$ with $i,j = e$ or g. Such equations are not easy to deal with, other than numerically, and we introduce now some approximations to simplify them.

5.3.3. Approximations leading to a Kramers-Fokker-Planck equation

We begin by introducing two small parameters characterizing atomic motion

$$\epsilon_1 = \frac{\hbar k_L}{\Delta p} = \frac{\text{recoil momentum}}{\text{momentum spread}} \tag{5.72}$$

$$\epsilon_2 = \frac{k_L\Delta p}{M\Gamma} = \frac{\text{Doppler effect}}{\text{natural width}} \tag{5.73}$$

Condition $\epsilon_1 \ll 1$ means also that the atomic coherence length $\xi_A = \hbar/\Delta p$ is small compared to the laser wavelength k_L^{-1}, which is equivalent to the localization assumption (2.28) introduced above and defining the semiclassical limit. Conditions $\epsilon_2 \ll 1$ means that the atomic velocity has been already damped enough (by laser cooling), so that one can treat the Doppler effect perturbatively. Such a condition is also equivalent to the localization assumption in momentum space, introduced above in

(2.21). Actually, near the Doppler cooling limit, we have, according to (5.67), $\Delta p^2/2M \sim \hbar\Gamma$, which leads to

$$\epsilon_1 \sim \epsilon_2 \sim \sqrt{E_R/\hbar\Gamma} \tag{5.74}$$

By expanding the generalized optical Bloch equations in powers of ϵ_1 and ϵ_2, it is then possible to replace these finite difference equations by coupled partial differential equations, easier to deal with. Another important point is that, at order 0 in ϵ_1 and ϵ_2, the Wigner function (5.69) does not evolve whereas all other variables vary with a time scale on the order of Γ^{-1}. This means that, in the limit of zero photon momentum ($k_L = 0$), photon-atom interactions cannot change the position or the velocity of the atom. This means also, since $\epsilon_1, \epsilon_2 \ll 1$, that there is a slow variable in the problem, $f(r,p)$, in terms of which all other variables can be adiabatically eliminated, leading to a single reduced evolution equation for $f(\mathbf{r},\mathbf{p})$, which is a quantum kinetic equation describing atomic motion.

Such a general procedure has been followed in several papers [25–28], and leads to a Kramers-Fokker-Planck equation for $f(\mathbf{r},\mathbf{p})$. The advantage of the treatment presented in Refs. 17 and 23 is that it uses an operatorial method not limited to two-level atoms, leading for the diffusion and friction coefficients, to general expressions with a more transparent structure in terms of two-time averages of the Heisenberg force operators. It is then possible to prove the equivalence of the results obtained in the Schrödinger and in the Heisenberg pictures and to get new physical insights in the friction coefficient and in the equilibrium temperature.

5.3.4. Physical discussion

The equation of motion of $f(\mathbf{r},\mathbf{p})$, derived in Ref. 17 has the following form

$$\begin{aligned}\frac{\partial f}{\partial t} = &-\frac{\mathbf{p}}{M}\cdot\frac{\partial f}{\partial \mathbf{r}} - \boldsymbol{\mathcal{F}}(\mathbf{r})\cdot\frac{\partial f}{\partial \mathbf{p}} \\ &+ \sum_{i,j=x,y,z} \frac{\partial^2 f}{\partial p_i \partial p_j}\left[D^{ij}_{\text{las}}(\mathbf{r}) + D^{ij}_{\text{vac}}(\mathbf{r})\right] \\ &+ \sum_{i,j=x,y,z} \gamma_{ij}(\mathbf{r})\frac{\partial}{\partial p_i}(p_j f) \\ &+ \text{Terms in } \partial^2 f/\partial p_i \partial r_j\end{aligned} \tag{5.75}$$

The term $-(\mathbf{p}/M).(\partial f/\partial \mathbf{r})$, which is of order 0 in ϵ_1 and ϵ_2, describes the free flight of the atom. The next term $-(\partial f/\partial \mathbf{p}).\boldsymbol{\mathcal{F}}(\mathbf{r})$, which is of

order 1 in ϵ_1, describes the drift in momentum of the Wigner function due to the mean radiative force $\mathcal{F}$, studied in Section 3 for an atom at rest in $\mathbf{r}$. The remaining terms are all of order 2 in ϵ_1 and ϵ_2, the last one (in $\partial^2 f/\partial p_i \partial r_j$) being negligible in most cases.

The terms in $\partial^2 f/\partial p_i \partial p_j$ describe momentum diffusion. D^{ij}_{las} and D^{ij}_{vac} are diffusion tensors given by equations similar to (5.41) and (5.42), each of the two $\delta\mathbf{F}$ operators being replaced by its i or j component. This shows that the two diffusion coefficients D_{las} and D_{vac} introduced in Subsection 5.2.2 are just the traces of the diffusion tensors appearing in (5.75).

The term in $\partial(p_j f)/\partial p_i$, with

$$\gamma_{ij}(\mathbf{r}) = \frac{i\hbar}{M} \int_0^\infty \tau \, \mathrm{d}\tau \, \langle [F_{Li}(\mathbf{r},\tau), F_{Lj}(\mathbf{r},0)] \rangle_{st} \tag{5.76}$$

where $\mathbf{F}_L = -\boldsymbol{\nabla} V_{AL}$ is the force operator associated with V_{AL} (first term of (2.16)), and where the average value is taken in the steady state of an atom at rest in $\mathbf{r}$, describes the friction. Combining this term with the term in $\partial f/\partial \mathbf{p}$ indeed gives

$$\sum_i \frac{\partial}{\partial p_i} \left[\mathcal{F}_i(\mathbf{r}) - \sum_j \gamma_{ij} p_j \right] f(\mathbf{r},\mathbf{p}) \tag{5.77}$$

so that $-\sum_j \gamma_{ij} p_j$ appears as a friction force, linear in p, correcting the force $\mathcal{F}_i$ obtained in Chapter 3 for an atom at rest in $\mathbf{r}$. In order to interpret the expression (5.76) of the friction tensor γ_{ij}, we first recall a well known result of linear response theory[29]. If a physical system $\mathcal{S}$ is in a stationary state σ_{eq}, and if it is perturbed by $V(t) = -\lambda(t)M$, where $\lambda(t)$ is a classical function of t and M an observable of $\mathcal{S}$, then, the mean value, at time t, of another observable N of $\mathcal{S}$ is given, to order 1 in λ, by

$$\langle N(t) \rangle = \langle N \rangle_{eq} + \int_{-\infty}^{+\infty} \mathrm{d}t' \, \chi_{NM}(t-t') \, \lambda(t') \tag{5.78}$$

where $\chi_{NM}(\tau)$ is a linear response function equal to

$$\chi_{NM}(\tau) = \frac{i}{\hbar} \theta(\tau) \, \langle [N(\tau), M(0)] \rangle_{eq} \tag{5.79}$$

In (5.79), $\theta(\tau)$ is the Heaviside function (equal to 1 for $\tau > 0$ and to 0 for $\tau < 0$) and $N(\tau)$ and $M(0)$ are free Heisenberg operators (evaluated in the absence of V). The fact that the mean value of a commutator appears in the expression (5.76) of the friction tensor then suggests to interpret γ_{ij} as a linear response function. More precisely, the interaction hamiltonian between the laser and a moving atom can be written

$$-\mathbf{d}\cdot\mathbf{E}_L\left(\mathbf{r}+\frac{\mathbf{p}}{M}(t-t_0)\right)\simeq -\mathbf{d}\cdot\mathbf{E}_L(\mathbf{r})+\sum_{j=x,y,z}-\frac{p_j}{M}(t-t_0)\nabla_j\mathbf{d}\cdot\mathbf{E}_L(\mathbf{r}) \tag{5.80}$$

The last term, where we recognize $F_{Lj}=\nabla_j(\mathbf{d}.\mathbf{E}_L)$, can be considered as a perturbation due to atomic motion and can be written as

$$-\sum_{j=x,y,z}\frac{p_j}{M}(t-t_\circ)F_{Lj}(\mathbf{r}) \tag{5.81}$$

i.e. as a sum of terms analogous to $V(t)$, with $\lambda(t)=-(t_0-t)P_j/M$ and $M=F_{Lj}$. It follows then from (5.78) that the mean value of $N=F_{Li}(\mathbf{r})$ at time t_0 is, to order1 in p, equal to

$$\begin{aligned}\langle F_{Li}(\mathbf{r},\mathbf{p},t_0)\rangle =&\langle F_{Li}(\mathbf{r},\mathbf{p}=\mathbf{0},t_0)\rangle\\ &+\sum_j\int_{-\infty}^{+\infty}\mathrm{dt}'\left[\chi_{F_{Li}F_{Lj}}(t_0-t')\right]\left[-(t_0-t')\frac{p_j}{M}\right]\end{aligned} \tag{5.82}$$

where we have put $t=t_0$. Using (5.79), one can then show that the last term of (5.82) can be written $-\sum_j\gamma_{ij}p_j$ and coincides with the friction force found in (5.77). The friction force thus appears as the linear response of the force to the perturbation due to atomic motion.

To summarize the results of this section, we see that there is a close analogy between atomic motion in laser light and Brownian motion in a potential well. Starting from the generalized optical Bloch equations, it is possible to derive for the atomic Wigner function, a kinetic equation quite analogous to the Kramers- Fokker-Planck equation (5.30) and to get simple physical interpretations for the diffusion and friction tensors in terms of correlation functions and linear susceptibilities. Finally, the equilibrium temperature can be written, according to (5.64) and to the results of this subsection, as

$$k_BT\sim\frac{D}{M\gamma}\sim\frac{\int_0^\infty\mathrm{d}\tau\,\langle\{\delta F(\tau)\delta F(0)+\delta F(0)\delta F(\tau)\}\rangle}{\frac{i}{\hbar}\int_0^\infty\tau\,\mathrm{d}\tau\,\langle\{\delta F(\tau)\delta F(0)-\delta F(0)\delta F(\tau)\}\rangle} \tag{5.83}$$

If the mean values of the anticommutator and commutator appearing respectively in the numerator and denominator of (5.83) are of the same order, we predict that

$$k_B T \sim \frac{\hbar}{\langle \tau \rangle} \tag{5.84}$$

where $\langle \tau \rangle$ is on the order of the correlation time of the Langevin force operator $\delta \mathbf{F}$. We have seen in Subsection 5.2.2 that the correlation function of $\delta \mathbf{F}$ is proportional to the correlation function of the atomic dipole moment d_z, so that $\langle \tau \rangle$ is on the order of an internal atomic time T_{int}. It follows that

$$k_B T \sim \frac{\hbar}{T_{\text{int}}} \tag{5.85}$$

For a two-level atom, $T_{\text{int}} \sim \Gamma^{-1}$, and we find again in (5.85) the Doppler limit (5.67). Equation (5.85) suggests that much lower temperatures can be reached if internal times much longer than Γ^{-1} exist. Examples of such situations will be given in the second part of this course.

6. Basic physical processes in the perturbative limit

6.1. Introduction

At low saturation ($s \ll 1$), i.e. at low intensity or large detuning, photon-atom interactions can be analyzed perturbatively in terms of elementary absorption and emission processes. For an atom in a plane wave, such an analysis, combined with the basic conservation laws, provides a simple interpretation of the main features of radiative forces : mean value, velocity dependence, fluctuations (see Sections 3.3, 4.1 and Subsection 5.2.3).

The situation becomes more complicated when the laser wave is a superposition of several plane waves. Because of the phase relations which exist between these waves, their contributions cannot be added independently. There are interference effects which make, for example, atomic motion in a laser standing wave more difficult to analyze than in a plane wave. An example of such difficulties is given by the very intriguing behaviour of an atom put in a node of a standing wave. In such a place, there is no light and no photon absorption, so that one would expect the atomic momentum diffusion coefficient D to vanish. However, such a naïve prediction is not confirmed by the calculation of