II.1 Definition of the Static Condition - The sources

The source of the electrostatic field is the charge density, $\rho(x)$. Under the static condition $\rho$ is independent of time. There are two possible situations where this can occur:

1. All charges are fixed in space
2. There is a steady current.

The first possibility is obvious. The second case implies that the flow of charge allows neither a buildup nor depletion of charge in any local region, so that although charges are moving, the local density of charge is time-independent.

The static condition then implies a constraint on the current density $J$ through conservation of charge. Mathematically, the conservation law is expressed through the continuity equation.

Consider a closed volume $V$ containing some charge $Q$, which is flowing in space

\[ \frac{dQ}{dt} = -\oint_S J \cdot da, \]

The rate of change $Q$ inside $V$ is the negative of the flux of current out of the volume,
where $s$ is the surface bounding $V$. The total charge in $V$ is $Q(t) = \int_V \rho(x,t) \, d^3x$, by definition. Thus, using the divergence theorem,

$$\frac{d}{dt} \int_V \rho(x,t) \, d^3x = -\int_V \nabla \cdot J \, d^3x.$$  

Assuming the volume is fixed in space, and since this must be true for an arbitrary $V$, the equality holds for the integrands,

$$\frac{d\rho}{dt} = -\nabla \cdot J. \quad \text{Continuity Equation}$$

The “static” condition $\partial \rho / \partial t = 0$ then implies, $\nabla \cdot J = 0$. Thus either $J=0$ (fixed charges), or the current is steady (no divergence). The latter case also encapsulates magnetostatics the topic of our next lecture.

II.2 Field Equations

Under the static condition, Maxwell’s equation read,

$$\nabla \cdot E = 4\pi \rho \quad \nabla \cdot B = 0$$

$$\nabla \times E = 0 \quad \nabla \times B = (4\pi / c)J$$

The field is completely defined by its divergence and curl, plus the boundary conditions on some enclosing surface (Helmholtz theorem). Unless stated otherwise, this condition will be that the fields go to zero on a surface at “infinity”.

We consider here the electrostatic field. The differential equations describe local relations between $E$ and $\rho$. The integral theorems give us the global relations,

- Gauss’s Law: $\int_V \nabla \cdot E \, d^3x = 4\pi \int_V \rho d^3x \Rightarrow \oint_S E \cdot da = 4\pi Q_{enc}$,  
  The flux of the electric field out of a closed surface is proportional to the enclosed charge.

- $\int_s (\nabla \times E) \cdot da = 0 \Rightarrow \oint_c E \cdot dl = 0$,  
  The circulation integral of $E$ is zero. This implies that electrostatic forces along are “conservative” and derivable from a potential energy $F = -\nabla U$. We define, the “electrostatic potential” $\phi = U/q$ (for a point charge) so that,

$$E = -\nabla \phi \Rightarrow \phi(x_2) - \phi(x_1) = -\int_{x_1}^{x_2} E \cdot dl \quad \text{(independent of path)}$$
II.3 Energy and Electrostatic fields

II.3.a Energy in an External Field

Suppose we have an electric field $E$ produced by some “external” sources. How much work is done by the field to move a charge from point $a$ to $b$?

$$W = \int_{a}^{b} F \cdot dl = q \int_{a}^{b} E \cdot dl = q(\phi(a) - \phi(b)) = -\Delta U_{ba}.$$

The work done is, by definition, the negative of the change in the potential energy going from $b$ to $a$. Since only changes in potential energy are related to the physical work, the zero of potential is arbitrary. This point is referred to as “ground”. Thus, relative to ground, the potential energy of a charge distribution in an external field is

$$U(x) = q \phi_{\text{external}}(x) = \int d^3x \rho(x) \phi_{\text{external}}(x).$$

II.3.b Energy of Configuration

Where does the potential energy come from? It is equal to the work necessary to “create” the external field in the first place. This in turn is equal to the work one would do to assemble a charge distribution, the is the source of the field. One imagines building up this distribution charge by charge, bringing in all charges from infinity (taken as “ground”).

The first charge from infinity requires no work. Then the second charge must be brought in with a force applied to counter the field of the first. Then the third against one and two, etc. The total work is then,

$$W_{\text{configure}} = q_1 \frac{q_2}{r_{12}} + q_3 \left( \frac{q_1}{r_{13}} + \frac{q_2}{r_{23}} \right).$$
For a collection of $N$ charges we then have,

$$W_{\text{configure}} = \frac{1}{2} \sum_{i \neq j}^{N} \frac{q_i q_j}{r_{ij}} = \frac{1}{2} \sum_{i}^{N} q_i \sum_{j \neq i}^{N} \frac{q_j}{r_{ij}} = \frac{1}{2} \sum_{i}^{N} q_i \phi(x_i) = \frac{1}{2} \int d^3 x \rho(x) \phi(x) = U(x).$$

The factor of $1/2$ in front of the sum avoids double counting. Note, that in contrast in Sec. II.3.a which represents the potential energy of a charge in an external field, this is the potential energy of the charge distribution itself. They are distinguished by the factor of $1/2$.

II.3.c Energy in the Electrostatic Field

Where is the energy of configuration stored? One way to do the bookkeeping is to consider the potential energy as being stored in the field itself! At the moment there is no real compelling reason to do so, we can equally well think of this as the potential energy of the charges. However, we will see in the dynamic case when the field is radiated in waves, if we are to preserve the concept of conservation of energy we must allow the fields themselves to posses energy. They take on a reality of their own.

Let us substitute for the charge in favor of the field from Gauss’s Law,

$$\rho = \frac{\nabla \cdot E}{4\pi} \Rightarrow U = \frac{1}{8\pi} \int d^3 x (\nabla \cdot \phi)(x).$$

We can then do integration by parts to move the derivative from $E$ onto $\phi$:

$$\nabla \cdot (\phi E) = \phi \nabla \cdot E + E \cdot \nabla \phi$$

$$\Rightarrow \phi \nabla \cdot E = -\nabla \phi \cdot E + \nabla \cdot (\phi E) = E \cdot \nabla \phi + \nabla \cdot (\phi E).$$

Thus,

$$U = \frac{1}{8\pi} \int_{V} d^3 x \cdot E + \frac{1}{8\pi} \int_{\partial V} \nabla \cdot (\phi E) d^2 x$$

$$= \frac{1}{8\pi} \int_{V} d^3 x |E|^2 + \frac{1}{8\pi} \int_{\partial S} \phi E \cdot da$$

The final integral is the “surface” term, bounding the volume. If we consider all of space, the boundary condition is that $\phi E \rightarrow 0$ at $\infty$. Thus we obtain an expression for the energy in the electric field:

$$U = \frac{1}{8\pi} \int d^3 x E \cdot E$$

Note for a point charge $U$ is infinite! This is because of the singularity - It takes an infinite energy to concentrate all of the charge into a point.
II.4 Finding the Electrostatic Field from the Sources

II.4a Gauss’s Law:

Though Gauss’ Law is universally true, it is not always useful for finding $E$ directly. However, under very special symmetric cases nothing is more powerful than Gauss’s Law. If we know a priori that $E \cdot \hat{n}$ is constant on some surface $S$, then,

$$\oint_S E \cdot da = (E \cdot \hat{n}) \oint_S da = E|_S A = 4\pi Q_{\text{enc}} \Rightarrow E|_S = \frac{4\pi Q_{\text{enc}}}{A}.$$ 

Examples:

- **Spherical symmetry, $\rho(r)$**: Charge density function only of radial distance $r$.

  We know that the magnitude of $E$ can only depend on $r$. Furthermore, since $\nabla \times E = 0$, we know that the direction of $E$ must point radially $\Rightarrow E(x) = E(r)\hat{r}$. Thus, $E \cdot \hat{n}$ is constant on surface of a sphere. This imaginary surface is known as a Gaussian surface,

  $$\oint_S E \cdot da = E(r)4\pi r^2 = 4\pi Q_{\text{enc}}(r) \Rightarrow E(r) = \frac{Q_{\text{enc}}(r)}{r^2}, \quad Q_{\text{enc}}(r) = Q_{\text{total}}, \quad r \geq R$$

  Thus, symmetry implies that the field at $r$ is nothing more than that of a point charge with all the enclosed charge at the origin.

- **Cylindrical symmetry, $\rho(r_c)$**, where $r_c$ is the” cylindrical” radius (no axial component). The cylinder is assumed to be “infinite” in length.

  Symmetry conditions demand that $E(x) = E(r_c)\hat{r}_c$. Thus, $E \cdot \hat{n}$ is constant on the surface of a cylinder.

  $$\oint_S E \cdot da = E(r)2\pi r L = 4\pi Q_{\text{enc}} \Rightarrow E(r) = \frac{2\pi Q_{\text{enc}}(r_c)}{r L} = \frac{2\pi \lambda(r_c)}{r_c}$$

  Outside the cylinder, we have the field of a line charge.

- **Planar symmetry, $\rho$** independent of two Cartesian coordinates (say $x$ and $y$) - infinite plane or slab with charge/area $\sigma$. From symmetry, the field cannot depend on $x$ or $y$ or the distance from the plane.

  $$\oint_S E \cdot da = 2E A = 4\pi \sigma A \Rightarrow E(x) = \begin{cases} 
2\pi \sigma \hat{z} & z > 0 \\
-2\pi \sigma \hat{z} & z < 0
\end{cases}$$
II.4a Poission’s and Laplace’s Equation

Generally there is insufficient symmetry to be able to solve for $E$ using Gauss’s law alone. Let us return to a general solution to the field equations. We have, $\nabla \times E = 0 \Rightarrow E = -\nabla \phi$. Plugging this into Gauss’s Law gives,

$$\nabla \cdot E = \nabla \cdot (-\nabla \phi) = 4\pi \rho \Rightarrow \nabla^2 \phi = -4\pi \rho.$$ 

The final form is known as Poisson’s Equation. If all of the sources are known, this equation, plus the boundary conditions determine $\phi$ as the sum of homogeneous and particular solution:

$$\phi(x) = \phi_{hom}(x) + \phi_{part}(x),$$

Where the homogeneous solution satisfies,

$$\nabla^2 \phi_{hom} = 0,$$ Laplace’s Equation.

The solution to Laplace’s equation can be chosen to satisfy the boundary conditions. In many situations, these maybe dictated by some macroscopic properties, e.g. a conducting surface over some region.

In the figure above the potential is set to $V$ by on a conducting surface by an external battery. Of course, at the microscopic level there are induced charge densities on the conductor, and these could be included in the source $\rho$. Fortunately, these details are unnecessary, and one can use the uniqueness theorems of PDE’s to solve for Laplace’s equation, solely from the boundary conditions. The techniques for doing so can be complex, but are well studied, and applicable to a wide variety of physical problems (acoustics, fluid mechanics, etc.). This is the topic of chapters 2 & 3 in Jackson, e.g. method of images, separation of variables, etc. Solving Laplace’s equation is, for some people, the central problem in electrostatics. We will not be studying it in this class, though you should have seen it in other courses.

We will focus our attention here to the potential of some given charge distribution $\rho$ in a region of space free from any other media. We then take as our boundary condition, $\phi \to 0$ at infinity, and the homogeneous solution can be taken to zero. We are left to solve for the particular solution. We can do so using the principle of superposition. Any distribution can be broken up into a collection of point charge. But what is the density of a point? In physics the mathematical object is the Dirac-Delta function. Suppose we have a unit point charge at position $x_0$. Then the charge density is given by,
\[ \rho_{\text{unit point}}(\mathbf{x}) = \delta^{(3)}(\mathbf{x} - \mathbf{x}_0) = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0). \]

The superscript (3) stands for three dimensions. In the case of Cartesian coordinates this separates as the product of familiar 1D delta functions, one for each coordinate - Be careful, this does not hold for curvilinear coordinate, e.g. spherical. The delta function satisfies,

\[ \int d^3x \delta^{(3)}(x - x_0) = 1, \quad \int d^3x f(x) \delta^{(3)}(x - x_0) = f(x_0). \]

Note that the dimensions of \( \delta^{(D)} \) is 1/D. Other properties are discussed in Jackson, Chapter 1.

We know from Coulomb’s Law that the potential at observation point \( \mathbf{x} \) given a unit point charge at \( \mathbf{x}' \) is

\[ \phi_{\text{unit point}}(\mathbf{x}) = \frac{1}{|\mathbf{x} - \mathbf{x}'|}. \]

This solution to is known as the Green’s function of Poisson’s equation,

\[ \nabla^2 G(\mathbf{x}; \mathbf{x}') = -4\pi \delta^{(3)}(\mathbf{x} - \mathbf{x}') \quad \Rightarrow \quad G(\mathbf{x}; \mathbf{x}') \equiv \phi_{\text{unit point at } \mathbf{x}'}(\mathbf{x}) = \frac{1}{|\mathbf{x} - \mathbf{x}'|}. \]

Although we wrote down this solution from our a priori knowledge of Coulomb’s Law, we can indeed solve for the Green’s function directly through, e.g. Fourier transforms.

The power of the Greens function is that we can now solve for the potential of an arbitrary charge distribution.

Because Poisson’s equation is linear, we can use the principle of superposition, and write the charge density as the superposition of point charges at positions \( \mathbf{x}' \), weighted by the strength of the density at that point, \( \rho(\mathbf{x}) = \int d^3x' \rho(\mathbf{x}') \delta^{(3)}(\mathbf{x} - \mathbf{x}'). \)

Thus we have the general solution,

\[ \phi(\mathbf{x}) = \int d^3x' \rho(\mathbf{x}') G(\mathbf{x}; \mathbf{x}') = \int d^3x' \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}. \]