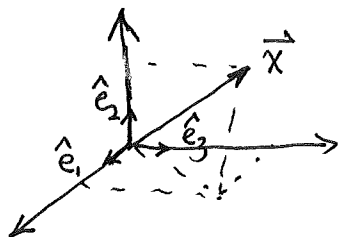


# Phys 511 Lecture #2: Vectors and Tensors

Basic Review Euclidean geometry and tensor notation

3D Cartesian space: position vector  $\vec{x}$



$$\vec{x} = x_1 \hat{e}_1 + x_2 \hat{e}_2 + x_3 \hat{e}_3 = \sum_{i=1}^3 x_i \hat{e}_i$$

$$\vec{x} \approx \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

"Representation in a basis"

Right handed coordinate basis  $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$

"Einstein Summation convention": Sum  $i=1 \rightarrow \text{dim}$  if in dx  $i$  is repeated  $\Rightarrow \vec{x} = x_i \hat{e}_i$

Inner product (dot)  $\Rightarrow$  Defines length and distance

$$\hat{e}_i \cdot \hat{e}_j = \begin{cases} 1 \\ 0 \end{cases} \text{ orthonormal vectors}$$

Notation  $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$  (Kronecker delta)

Linear operation  $\vec{x}_a \cdot (c_1 \vec{x}_b + c_2 \vec{x}_c) = c_1 (\vec{x}_a \cdot \vec{x}_b) + c_2 (\vec{x}_a \cdot \vec{x}_c)$

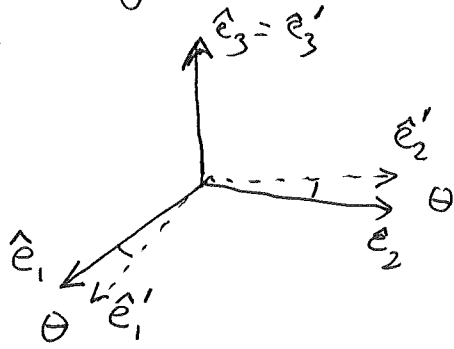
$$\Rightarrow \vec{x} \cdot \vec{x} = (x_i \hat{e}_i) \cdot (x_j \hat{e}_j) = x_i x_j (\hat{e}_i \cdot \hat{e}_j) = x_i x_j \delta_{ij} = x_i x_i$$

Note never reuse dummy indices

$$\Rightarrow \vec{x} \cdot \vec{x} = x_1^2 + x_2^2 + x_3^2 = |\vec{x}|^2 \text{ Length of } \vec{x} \text{ squared (by Pythagoras)}$$

Law of Cosines  $\Rightarrow \vec{x}_a \cdot \vec{x}_b = |\vec{x}_a| |\vec{x}_b| \cos \theta_{ab}$

Change of basis  $\Rightarrow$  Definition of vector



$$\begin{aligned} \hat{e}_1' &= \cos\theta \hat{e}_1 + \sin\theta \hat{e}_2 \\ \hat{e}_2' &= -\sin\theta \hat{e}_1 + \cos\theta \hat{e}_2 \\ \hat{e}_3' &= \hat{e}_3 \end{aligned}$$

example rotation of coordinate system

$$\hat{e}_i' = R_{ij} \hat{e}_j \quad R = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{ij} = \hat{e}_i' \cdot \hat{e}_j$$

$$\begin{aligned} \hat{e}_i' \cdot \hat{e}_j' &= (R_{ik} \hat{e}_k) \cdot (R_{jl} \hat{e}_l) = R_{ik} R_{jl} \hat{e}_k \cdot \hat{e}_l = R_{ik} R_{jl} \delta_{kl} \\ &= R_{ik} R_{jk} = (R)_{ik} (R^T)_{kj} = (R R^T)_{ij} \end{aligned}$$

But  $\hat{e}_i' \cdot \hat{e}_j' = \delta_{ij}$  (lengths are preserved in rotation)  
 $= (\mathbb{1})_{ij}$

$$\Rightarrow R R^T = \mathbb{1} \Rightarrow \boxed{R^{-1} = R^T}$$

Orthogonal transformation  
 $\det(R) = \pm 1$

Two kinds of orthogonal transformations

• "Proper" (rotations)  
 $\det(R) = +1$

• "Improper" (reflections)  
 example  $\hat{e}_1' = \hat{e}_1, \hat{e}_2' = \hat{e}_2, \hat{e}_3' = -\hat{e}_3$

$$\det(R) = -1$$

## Transformation of components

L2.2

Same physical position, two different representations

$$\vec{x} = x_i \hat{e}_i = x'_j \hat{e}'_j = x'_j R_{jk} \hat{e}_k = (R^T_{kj} x'_j) \hat{e}_k$$

$$\Rightarrow \underline{\tilde{x}} = \underline{\tilde{R}^T} \underline{\tilde{x}'} \Rightarrow \underline{\tilde{x}'} = (\underline{\tilde{R}^T})^{-1} \underline{\tilde{x}}$$

For Euclidean  $(\underline{\tilde{R}^T})^{-1} = \underline{\tilde{R}} \Rightarrow \underline{\tilde{x}'} = \underline{\tilde{R}} \underline{\tilde{x}}$

$$\Rightarrow \boxed{x'_i = R_{ij} x_j}$$

Coordinates ~~and~~ and basis vectors transform the same (not true in general)

## Abstract definition of a vector

Geometrical object  $\vec{V} = \sum_{i=1}^3 V_i \hat{e}_i = V_i \hat{e}_i$

Define vector such that components of  $\vec{V}$  transform as the components of  $\vec{x}$  under orthogonal transformation

$$\boxed{V'_i = R_{ij} V_j}$$

Example: momentum  $\vec{p} = m \frac{d\vec{x}}{dt}$ ,  $m, t$  scalars

# Cross-products and pseudo-vectors

L 2.3

Given an orthonormal basis we define:

$$\hat{e}_i \times \hat{e}_j = \epsilon_{ijk} \hat{e}_k \quad : \quad \text{Vector cross product}$$

• Levi-Civita symbol  $\epsilon_{ijk} = \begin{cases} 1 & i \neq j \neq k \\ -1 & \text{same} \\ 0 & \text{otherwise} \end{cases}$  and/or  
 $i, j, k = \text{cyclic } 1, 2, 3$   
non cyclic

$$\left. \begin{aligned} \epsilon_{1,2,3} = \epsilon_{3,1,2} = \epsilon_{2,3,1} = 1 \\ \epsilon_{1,3,2} = \epsilon_{2,1,3} = \epsilon_{3,2,1} = -1 \end{aligned} \right\} \begin{array}{l} \epsilon_{ijk} \text{ is antisymmetric} \\ \text{w.r.t exchange of any two indices} \end{array}$$

## • Important relations

$$\epsilon_{ijk} U_i V_j W_k = \det \begin{bmatrix} U_1 & U_2 & U_3 \\ V_1 & V_2 & V_3 \\ W_1 & W_2 & W_3 \end{bmatrix}$$

$$\Rightarrow \vec{C} = \vec{A} \times \vec{B} = (\hat{e}_i \times \hat{e}_j) A_i B_j = \epsilon_{ijk} \hat{e}_k A_i B_j$$

$$\vec{C} = \epsilon_{ijk} A_j B_k \hat{e}_i$$

$$\Rightarrow (\vec{A} \times \vec{B})_i = \epsilon_{ijk} A_j B_k$$

Using determinant:  $\vec{A} \times \vec{B} = \det \begin{bmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{bmatrix}$

• Contracting on any two indices

$$\epsilon_{ijk} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$$

Using tensor notation to prove vector identities

(L2.4)

e.g.  $\vec{D} = \vec{A} \times (\vec{B} \times \vec{C})$

$$D_i = \epsilon_{ijk} A_j (\vec{B} \times \vec{C})_k = \epsilon_{ijk} \epsilon_{klm} A_j B_l C_m$$

$$= [\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}] A_j B_l C_m = A_j B_l C_j - A_j B_j C_l$$

$$= B_l (\vec{A} \cdot \vec{C}) - C_l (\vec{A} \cdot \vec{B})$$

$$\Rightarrow \boxed{\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B} (\vec{A} \cdot \vec{C}) - \vec{C} (\vec{A} \cdot \vec{B})}$$

Is the cross product between two vectors a vector?

Let  $\vec{w} = \vec{u} \times \vec{v} \Rightarrow w_i = \epsilon_{ijk} u_j v_k$

In another orthogonal coordinate system:  $w'_i = \epsilon_{ijk}$

$$w'_i = \epsilon_{ijk} R_{jl} R_{km} u_l v_m = \epsilon_{nyk} \delta_{in} R_{je} R_{km} u_e v_m$$

Insert  $\delta_{in} = R_{ip} R_{np}$

$$\Rightarrow w'_i = R_{ip} \underbrace{\epsilon_{nyk} R_{np} R_{je} R_{km}}_{\det[\tilde{R}] \epsilon_{pem}} u_e v_m$$

$$\Rightarrow w'_i = \det[\tilde{R}] R_{ip} (\vec{u} \times \vec{v})_p = \det[\tilde{R}] R_{ip} w_p$$

$$\Rightarrow \boxed{\vec{w} \text{ is a pseudo vector}}$$

# Tensors:

(25)

Consider rigid body rotation:

$$\vec{L} = \sum_{\alpha} \vec{x}_{\alpha} \times (m_{\alpha} \vec{v}_{\alpha}) = \sum_{\alpha} m_{\alpha} \vec{x}_{\alpha} \times (\vec{\omega} \times \vec{x}_{\alpha})$$

(particles)

$$= \sum_{\alpha} m_{\alpha} (r_{\alpha}^2 \vec{\omega} - \vec{x}_{\alpha} (\vec{x}_{\alpha} \cdot \vec{\omega})) = \underline{\underline{I}} \cdot \vec{\omega}$$

$$\underline{\underline{I}} = \sum_{\alpha} m_{\alpha} (r_{\alpha}^2 \underline{\underline{1}} - \underbrace{\vec{x}_{\alpha} \vec{x}_{\alpha}}_{\text{"outer product"}})$$

unit tensor

In components:  $L_i = I_{ij} \omega_j$

$$I_{ij} = \sum_{\alpha} m_{\alpha} (r_{\alpha}^2 \delta_{ij} - x_i^{(\alpha)} x_j^{(\alpha)})$$

In a new coordinate system  $\delta_{ij} \rightarrow \delta'_{ij}$   $x_i^{(\alpha)} \rightarrow x_i'^{(\alpha)}$

$$I'_{ij} = \sum_{\alpha} m_{\alpha} (r_{\alpha}^2 \delta'_{ij} - R_{ik} x_n^{(\alpha)} R_{jl} x_l^{(\alpha)})$$

$$\Rightarrow \boxed{I'_{ij} = R_{ik} R_{jl} I_{kl}}$$

Definition: A rank- $n$  tensor is defined such that its ~~comp~~  $n$  components transform as

$$\boxed{T_{i_1 i_2 \dots i_n} = R_{i_1 j_1} R_{i_2 j_2} \dots R_{i_n j_n} T_{j_1 j_2 \dots j_n}}$$

• Example of a tensor: The outer product of  $n$  vectors (tensor product) between  $n$  vectors is a rank- $n$  tensor

$$T_{ijk} = U_i V_j W_k \quad (\text{obvious})$$

• Summing (contracting) on any indices (tracing over indices) reduces the rank by two

eg.  $T_{iik} = (\vec{U} \cdot \vec{V}) U_k \iff$  rank 1 tensor (vector)

$M_{ii} = M_{11} + M_{22} + M_{33} = \text{Tr}(M)$  : Trace operator in matrix algebra

• Relation between cross-product and antisymmetric tensor

Let  $T_{ij} = \epsilon_{ijk} t_k \implies T \approx \begin{bmatrix} 0 & t_3 & -t_2 \\ -t_3 & 0 & t_1 \\ t_2 & -t_1 & 0 \end{bmatrix}$

antisymmetric rank-2 tensor  $\equiv$  components of a cross product

$$t_k = (\vec{V} \times \vec{W})_k = \epsilon_{k\ell m} V_\ell W_m$$

$$\implies T_{ij} = V_i W_j - V_j W_i \quad (\text{Anti-symmetrized outer product})$$

$t_k$  and  $T_{ij}$  are "duals"

$t_i = \frac{1}{2} \epsilon_{ijk} T_{jk}$