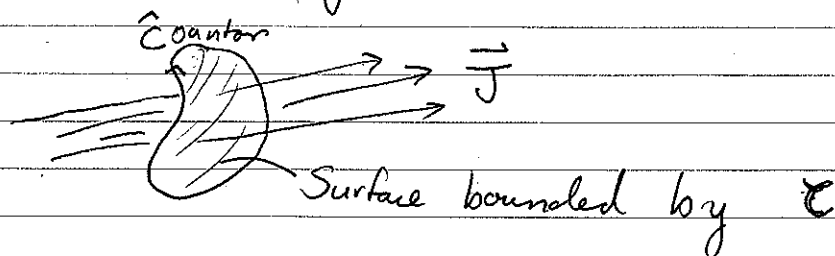


Physics 511 - Lecture #8

Magnetostatics II

Ampère's Law: Integral Form



Flux of \vec{J} through $S \Rightarrow \int_S \vec{J} \cdot d\vec{a} = I_{enc}$
= Current enclosed by C

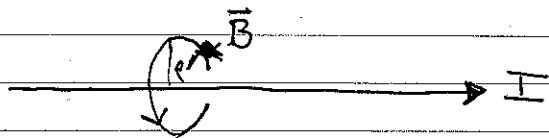
$$\Rightarrow \int \vec{J} \cdot d\vec{a} = \frac{c}{4\pi} \int (\nabla \times \vec{B}) \cdot d\vec{a} \xrightarrow{\text{Stokes theorem}} \frac{c}{4\pi} \oint_C \vec{B} \cdot d\vec{l}$$

$$\Rightarrow \boxed{\oint_C \vec{B} \cdot d\vec{l} = \frac{4\pi}{c} I_{enc}}$$

Remember Right Hand Rule

Use Integral form of Ampère's Law like Gauss's Law to find \vec{B} for highly symmetric distributions

- Most basic example: Infinite line current

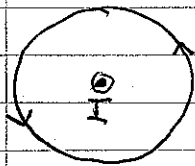


Head on

Symmetry

$$\vec{B} = B(\rho) \vec{e}_\phi$$

radial distance from axis

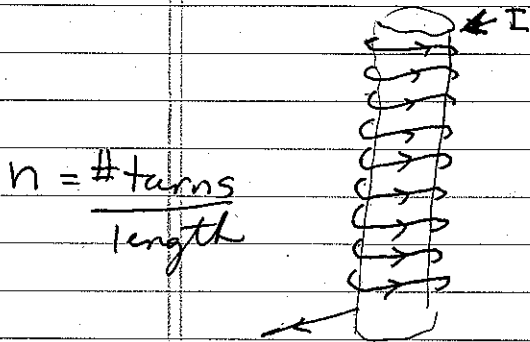


"Amperian loop"

$$\oint_C \vec{B} \cdot d\vec{l} = B(\rho) \oint_C dl = (2\pi\rho) B(\rho) = \frac{4\pi}{c} I$$

$$\Rightarrow \boxed{\vec{B} = \frac{2I}{c\rho} \vec{e}_\phi}$$

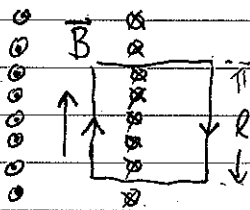
• Eg Infinite Solenoid



Using Biot-Savart we can prove that B is uniform inside, zero outside

$\Rightarrow \vec{B} = B_0 \hat{e}_z$ inside
^ symmetry

Cross section

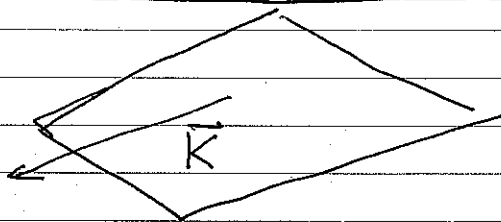


$\oint \vec{B} \cdot d\vec{l} = Bl = \frac{4\pi}{c} \int \vec{J} \cdot d\vec{a} = \frac{4\pi}{c} N_{\text{turns}} I$

$\Rightarrow \vec{B} = \frac{4\pi}{c} n I \hat{e}_z$ inside
 $= 0$ outside

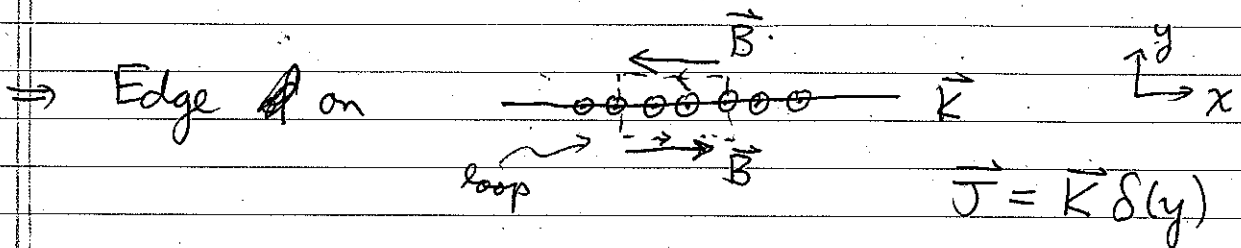
Note $nI = K$ (surface current)

• Eg Planar Sheet of current



Infinite $\Rightarrow |\vec{B}|$ independent of distance from plane

$\vec{B} \perp$ to \vec{K} and \perp to \vec{x}



$\oint \vec{B} \cdot d\vec{l} = 2\ell B = \frac{4\pi}{c} \int \vec{J} \cdot d\vec{a} = \frac{4\pi}{c} \int K dx = \frac{4\pi}{c} K\ell$

$\Rightarrow \boxed{B = \frac{2\pi}{c} K}$

Formal Solution in magnetostatics

Field equations: $\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \vec{B} = \vec{\nabla} \times \vec{A}$

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J} \Rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \frac{4\pi}{c} \vec{J}$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} = \frac{4\pi}{c} \vec{J}$$

Gauge invariance: Can always add an "irrotational"

vector field to \vec{A} without changing \vec{B}

$$\vec{A} \Rightarrow \vec{A} + \vec{\nabla} \Lambda$$

$$\vec{B} \Rightarrow \vec{\nabla} \times (\vec{A} + \vec{\nabla} \Lambda) = \vec{\nabla} \times \vec{A}$$

Gauge transformation

$$\vec{B} \Rightarrow \vec{B}$$

Fixing the gauge: In magnetostatics choose \vec{A}

such that $\vec{\nabla} \cdot \vec{A} = 0$ + Always possible

$$\boxed{\nabla^2 \vec{A} = -\frac{4\pi}{c} \vec{J}}$$
 Poisson's equation for each Cartesian component

Formal Solution

$$\boxed{\vec{A} = \frac{1}{c} \int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|}}$$

(assuming $\vec{J} \rightarrow 0$ at ∞)

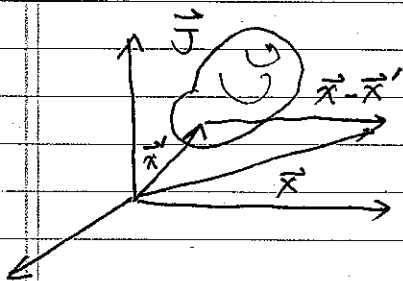
Careful:

This is a vector equation so generally, direction of $\vec{J}(\vec{x}')$ depends on \vec{x}'

Not do-able for most \vec{J}

Need approximations \Rightarrow Multipole expansion

Multipole Expansion in Magnetostatics



In "transverse-gauge"

$$\vec{A}(\vec{x}) = \frac{1}{c} \int \frac{\vec{J}(\vec{x}') d^3x'}{|\vec{x} - \vec{x}'|}$$

with $\vec{\nabla} \cdot \vec{J} = 0$, localized source

Taylor series: $\frac{1}{|\vec{x} - \vec{x}'|} = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} (\vec{x}' \cdot \vec{\nabla})^l \frac{1}{r}$

"Monopole Term" $\vec{A}^{(0)} = \frac{1}{cr} \int d^3x' \vec{J}(\vec{x}') = 0$

Proof: $\vec{\nabla}' \cdot (x'_i \vec{J}(\vec{x}')) = \partial'_k (x'_i J'_k(\vec{x}')) = \delta_{ik} J'_k + x'_i \underbrace{(\vec{\nabla}' \cdot \vec{J}')}_{\vec{\nabla}' \cdot \vec{J}'}$

$$\Rightarrow \int_V \vec{J}(\vec{x}') d^3x' = \int_V \vec{\nabla}' \cdot (\vec{x}' \vec{J}(\vec{x}')) d^3x = \oint_S (\vec{x}' \vec{J}(\vec{x}')) \cdot d\vec{a}$$

as $V \rightarrow \infty$ $\int_V \vec{J}(\vec{x}') d^3x' = 0$

"Dipole term": $\vec{A}^{(1)}(\vec{x}) = -\frac{1}{c} \int d^3x' \vec{J}(\vec{x}') (\vec{x}' \cdot \vec{\nabla}) \frac{1}{r}$

Tensor notation $A_i^{(1)}(\vec{x}) = \left(-\frac{1}{c} \int d^3x' J_i(\vec{x}') x'_k \right) \partial_k \frac{1}{r}$

Break up into symmetric/asymmetric pieces

$$J_i x'_k = \frac{1}{2} (J_i x'_k + J_k x'_i) + \frac{1}{2} (J_i x'_k - J_k x'_i)$$

First term integrates to zero:

$$\vec{\nabla} \cdot (x'_i x'_j \vec{J}) = x'_j J_i(\vec{x}') + x'_i J_j(\vec{x}')$$

$$\begin{aligned} \therefore A_i^{(1)} &= -\frac{1}{2c} \int d^3x' (x'_j J_i - x'_i J_j) \left(\partial_j \frac{1}{r} \right) \\ &= \frac{1}{2c} \int d^3x' \epsilon_{ijk} (\vec{x}' \times \vec{J}(\vec{x}'))_k \left(\partial_j \frac{1}{r} \right) \end{aligned}$$

$$A_i^{(1)} = \epsilon_{ijk} \left[\frac{1}{2c} \int d^3x' (\vec{x}' \times \vec{J}(\vec{x}')) \right]_k \partial_j \frac{1}{r}$$

$$\Rightarrow \boxed{\begin{aligned} \vec{A}^{(1)} &= -\vec{m} \times \vec{\nabla} \frac{1}{r} = \frac{\vec{m} \times \hat{r}}{r^2} \\ \vec{m} &= \frac{1}{2c} \int d^3x' \vec{x}' \times \vec{J}(\vec{x}') \quad \text{Magnetic dipole moment} \end{aligned}}$$

For a magnetic dipole:

$$\vec{B}^{(1)} = \vec{\nabla} \times \vec{A} = -\vec{\nabla} \times (\vec{m} \times \vec{\nabla} \frac{1}{r}) = -\epsilon_{ijk} \epsilon_{klp} \partial_j m_l \partial_p \frac{1}{r}$$

$$= -(\delta_{il} \delta_{jp} - \delta_{ip} \delta_{jl}) m_l \partial_j \partial_p \frac{1}{r}$$

$$= -m_i \left(\cancel{\partial_j^2 \frac{1}{r}} \right) + m_j \partial_j \partial_i \frac{1}{r} = \partial_i \vec{m} \cdot \vec{\nabla} \frac{1}{r}$$

0 $r \neq 0$

$$= -\partial_i \vec{m} \cdot \frac{\hat{r}}{r^2}$$

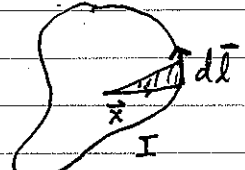
$$\therefore \boxed{\vec{B}^{(1)} = -\vec{\nabla} \left(\frac{\hat{r} \cdot \vec{m}}{r^2} \right)} \quad \begin{array}{l} \text{pseudo} \\ \text{potential} \end{array} \quad \text{"Field of dipole"}$$

Compare
to electric dipole:

$$\vec{E} = -\vec{\nabla} \left(\frac{\hat{r} \cdot \vec{p}}{r^2} \right)$$

Two important cases

(i) Planar current loop:



area swept out $dA = \frac{1}{2} \vec{r} \times d\vec{l}$

$$\vec{m} = \frac{1}{2c} \int d^3x \vec{x} \times \vec{J}(\vec{x}) = \frac{I}{2c} \oint \vec{x} \times d\vec{l}$$

$$= \frac{I}{c} \oint dA \hat{n} = \boxed{\frac{I}{c} \text{Area } \hat{n} = \vec{m}}$$

(ii) Point charge in orbit: $\vec{J}(\vec{x}) = q \vec{v}_p \delta^{(3)}(\vec{x} - \vec{x}_p(t))$

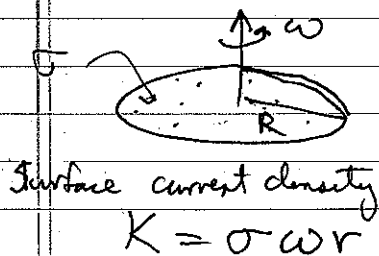
$$\vec{m} = \frac{1}{2c} \int d^3x \vec{x} \times \vec{J}(\vec{x}) = q \frac{\vec{x}_p \times \vec{v}_p}{2c}$$

$$\Rightarrow \boxed{\vec{m} = \frac{q}{2mc} \vec{L}_{\text{particle}}}$$

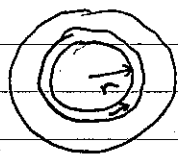
$\vec{L} =$ orbital angular momentum
 $\frac{q}{2mc} \equiv$ gyromagnetic ratio

Not exact since current not steady state. Approximate if $v \ll c$

Example: Rotating disk of surface charge



Break up into rings



$$dI = K(r) dr = \sigma \omega r dr$$

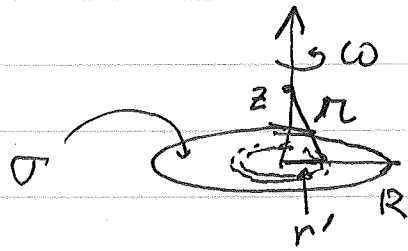
$$\Rightarrow d\vec{m} = \frac{dI}{c} \pi r^2 \hat{e}_z = \frac{1}{c} \sigma \omega \pi r^3 dr \hat{e}_z$$

$$\Rightarrow \vec{m} = \int_0^R d\vec{m}(r) \hat{e}_z = \frac{\pi \sigma \omega R^4}{4c} \hat{e}_z$$

Far away from disk $\vec{B} = \vec{\nabla} \cdot \left(\hat{r} \cdot \frac{\vec{m}}{r^2} \right) \quad r \gg R$

On z-axis $\vec{B}(z) \approx \frac{2m}{z^3} \hat{e}_z = \frac{\pi \sigma \omega R^4}{2c z^3} \hat{e}_z$ (Compare to limit of exact expression)

Magnetic field of a rotating disk of charge



Surface current density
 $K = \sigma v = \sigma \omega r$

$$dB_z(z) = \frac{2\pi r'^2}{(z^2 + r'^2)^{3/2}} dI \quad \leftarrow \text{Current in differential ring}$$

$$dI = K dr'$$

$$\Rightarrow dB_z(z) = \frac{2\pi\sigma\omega}{c} \frac{r'^3 dr'}{(z^2 + r'^2)^{3/2}}$$

$$\therefore B_z(z) = \frac{2\pi\sigma\omega}{c} \int_0^R \frac{r'^3 dr'}{(z^2 + r'^2)^{3/2}} = \frac{2\pi\sigma\omega}{c} \left(\frac{r'^2 + 2z^2}{\sqrt{r'^2 + z^2}} \right)_0^R$$

$$\Rightarrow B(z) = \frac{2\pi\sigma\omega}{c} \left(-2z + \frac{R^2 + 2z^2}{\sqrt{R^2 + z^2}} \right)$$

Limits

$$z \ll R \quad B(z) \rightarrow \frac{2\pi\sigma\omega R}{c} = \frac{2\pi K R}{c}$$

$$z \gg R \quad B(z) = \frac{2\pi\sigma\omega}{c} \left(-2z + \left(\frac{R^2}{z} + 2z \right) \left(1 + \frac{R^2}{z^2} \right)^{-1/2} \right)$$

$$\approx \frac{2\pi\sigma\omega}{c} \left(-2z + \left(\frac{R^2}{z} + 2z \right) \left(1 - \frac{R^2}{2z^2} + \frac{3}{8} \frac{R^4}{z^4} \right) \right)$$

$$= \frac{2\pi\sigma\omega}{c} \left(-2z + \frac{R^2}{z} + 2z - \frac{R^4}{2z^3} - \frac{R^2}{z} + \frac{3R^4}{4z^3} \right)$$

$$\Rightarrow B(z) = \frac{2m}{z^3}, \quad \text{magnetic dipole: } m = \pi\sigma\omega R^4/4c \quad \checkmark$$

Forces on currents in external fields

$$\vec{F} = \frac{1}{c} \int d^3x' \vec{J}(\vec{x}') \times \vec{B}(\vec{x}') \Rightarrow F_i = \frac{1}{c} \epsilon_{ijk} \int d^3x' J_j(\vec{x}') B_k(\vec{x}')$$

Distribution localized at origin \Rightarrow Expand $B_k(\vec{x}')$

$$B_k(\vec{x}') \cong B_k(0) + \vec{x}' \cdot \nabla B_k \Big|_{x=0} + \dots$$

$$\Rightarrow F_i = \frac{1}{c} \epsilon_{ijk} \left(\int d^3x' J_j(\vec{x}') B_k(0) + \epsilon_{jlp} m_p \partial_l B_k \Big|_{x=0} \right)$$

Aside

$$\epsilon_{ijk} \epsilon_{jlp} = \epsilon_{ikj} \epsilon_{jlp} = \delta_{il} \delta_{kp} - \delta_{ip} \delta_{kl}$$

$$\Rightarrow F_i = \epsilon_{ijk} m_k \partial_i B_k - m_i \partial_k B_k$$

$$\Rightarrow \vec{F} = \nabla(\vec{m} \cdot \vec{B}) - \vec{m} (\nabla \cdot \vec{B}) \Rightarrow \boxed{\vec{F} = \nabla(\vec{m} \cdot \vec{B})}$$

Potential energy: $\boxed{U = -\vec{m} \cdot \vec{B}}$

• Constant field: Torque $\vec{N} = \vec{m} \times \vec{B}$

• Gradient field: Trap

