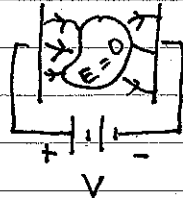


Faraday's Law and Quasi-statics - Lecture #9

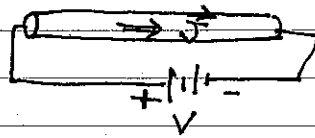
Beyond Statics

• Electro-statics



Electric field is zero inside the conductor

• Magnetostatics



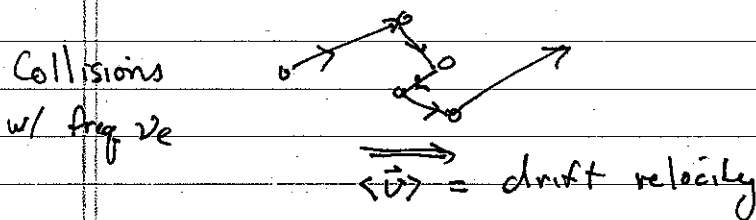
Current flowing in conductor \Rightarrow must be a force on charges

Since work is done on charges, must be due to \vec{E} -field

Phenomenological relation $\vec{J} = \sigma_c \vec{E}$ (Ohm's "Law")

$\sigma_c =$ conductivity (Generally tensor $\vec{J} = \underline{\underline{\sigma}}_c \cdot \vec{E}$)

Drude model: Free electron gas, collisions with defects



Macroscopic current

$$\vec{J} = -en_e \langle \vec{v} \rangle$$

\uparrow
density of electrons

Equation of motion: $m_e \frac{d\langle \vec{v} \rangle}{dt} = -e\vec{E} - \underbrace{(m_e \nu_e)}_{\text{Drag force}} \langle \vec{v} \rangle$

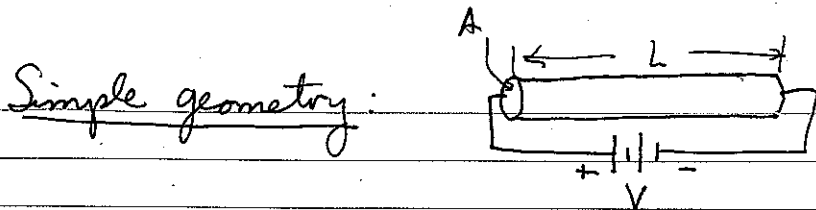
$$\Rightarrow \boxed{\frac{d\vec{J}}{dt} + \nu_e \vec{J} = \frac{\omega_p^2}{4\pi} \vec{E}}$$

$$\omega_p^2 = \frac{4\pi n_e e^2}{m_e} \quad (\text{Plasma frequency})$$

Generalized Ohm's Law

DC. / steady state limit \Rightarrow
 $T \gg \frac{1}{\nu_e}$

$$\vec{J} = \frac{\omega_p^2}{4\pi \nu_e} \vec{E} = \sigma_c \vec{E}; \quad \sigma_c = \frac{1}{4\pi \nu_e}$$



Ignoring fringe fields
 $E = \frac{V}{L}$, $J = \frac{I}{A}$

$$J = \sigma_c E \Rightarrow \frac{I}{A} = \sigma_c \frac{V}{L} \Rightarrow V = \left(\frac{L}{A \sigma_c} \right) I$$

$$\Rightarrow \boxed{V = IR} \quad \text{Resistance } R = \frac{L}{A} \rho_c \leftarrow \frac{1}{\sigma_c} : \text{resistivity}$$

Because of resistance, flowing charges lose energy

Ohmic heating: Rate at which work is done on charges

$$\vec{F}_i = q_i \vec{E}_i + q_i \frac{\vec{v}_i}{c} \times \vec{B}_i \Rightarrow \frac{dW}{dt} = \sum_i \vec{v}_i \cdot \vec{F}_i$$

$$\Rightarrow \frac{dW}{dt} = \sum_i q_i \vec{v}_i \cdot \vec{E}_i = \int d^3x \vec{J}(\vec{x}) \cdot \vec{E}(\vec{x})$$

$$(\vec{J}(\vec{x}) = \sum_i q_i \vec{v}_i \delta(\vec{x} - \vec{x}_i))$$

$$(= IV = I^2 R \text{ for simple geometry})$$

$$\Rightarrow \boxed{\vec{J} \cdot \vec{E} = \text{rate of energy dissipation/volume}}$$

EMF: If energy is being dissipated, then the battery is doing work

$$\oint_c \vec{F} \cdot d\vec{l} = \text{Work done to push charge around circuit}$$

$$\vec{F} = \vec{F}_{\text{battery}} + \vec{F}_{\text{electrostatic}}$$

$$\text{Define } \boxed{\mathcal{E} = \frac{1}{q} \oint \vec{F} \cdot d\vec{l} : \text{Work/charge (Historic name EMF) "Electromotive Force"} = \text{Voltage}}$$

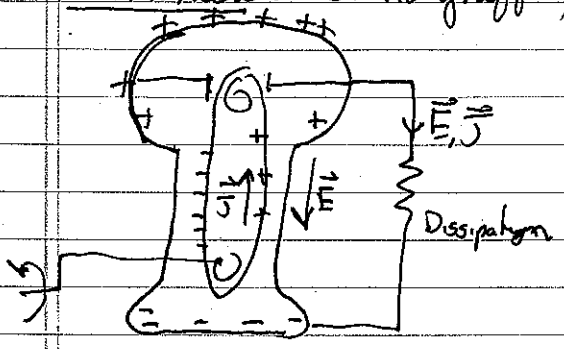
$$\mathcal{E} = \frac{1}{q} \oint \vec{F}_{\text{battery}} \cdot d\vec{l} + \frac{1}{q} \oint \vec{F}_{\text{electrostatic}} \cdot d\vec{l}$$

Since $\oint_c \vec{E} \cdot d\vec{l} = 0$

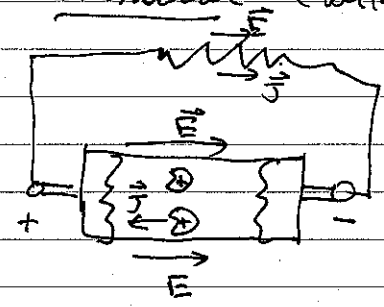
Electrostatic fields cannot provide EMF (conservative field)

Examples of EMF

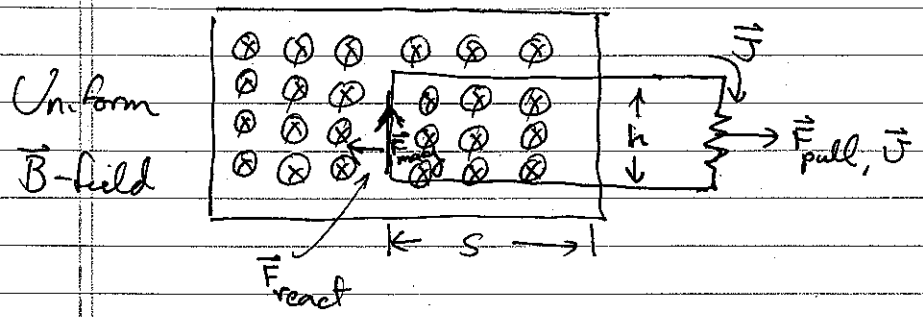
• Mechanical (Vandegriff)



• Chemical (battery)



"Motional" EMF : Lorentz Force



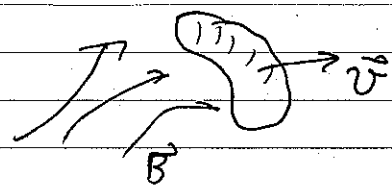
$|\vec{F}_{mag}| = q \frac{v}{c} B$
 (Lorentz force on electrons inside the wire as they are dragged through \vec{B})

$$\mathcal{E} = \oint \frac{\vec{F}}{q} \cdot d\vec{l} = \frac{v}{c} B h = \frac{1}{c} \left(-\frac{ds}{dt} \right) h B = -\frac{1}{c} \frac{d\Phi_B}{dt}$$

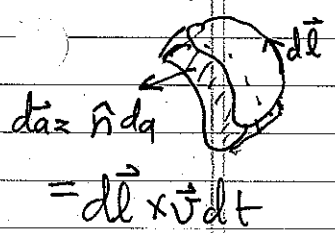
$$\mathcal{E} = -\frac{1}{c} \frac{d\Phi_B}{dt} \quad \Phi_B = \int \vec{B} \cdot d\vec{a} = (sh) B$$

General geometrical feature:

Loop moving through \vec{B} -field with velocity \vec{v} :



Loop sweeps out volume:



$$\oint \vec{B} \cdot d\vec{a} = 0 \Rightarrow \Phi_B(t+dt) - \Phi_B(t) = -\int_{sides} \vec{B} \cdot d\vec{a}$$

$$\Rightarrow d\Phi_B = -\oint dt \oint \vec{B} \cdot (d\vec{l} \times \vec{v}) = +\oint (\vec{B} \times \vec{v}) \cdot d\vec{l} dt$$

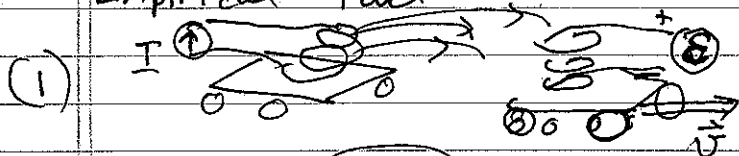
Modern side $\Rightarrow d\Phi_B = - \oint_C (\vec{v} \times \vec{B}) \cdot d\vec{l} dt$

$\Rightarrow \frac{d\Phi_B}{dt} = -c \oint_C \vec{E} \cdot d\vec{l} \Rightarrow \boxed{\mathcal{E}_{\text{motional}} = -\frac{1}{c} \frac{d\Phi}{dt}}$

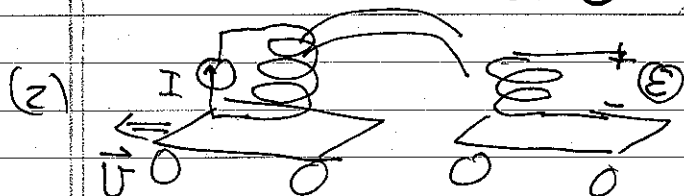
No new physics, just Lorentz force

Faraday's Universal Law of Induction

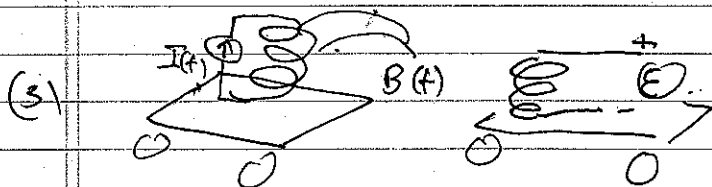
Empirical fact



Move coil through non-uniform \vec{B} -field



Move source of \vec{B} -field, keep coil fixed



Keep coil + \vec{B} -field fixed, ~~move~~ change $B(t)$ by changing $I(t)$

All give the same answer $\boxed{\mathcal{E} = -\frac{1}{c} \frac{d\Phi_B}{dt}}$

Lenz's Law

Modern understanding: Lorentz invariance (Einstein)

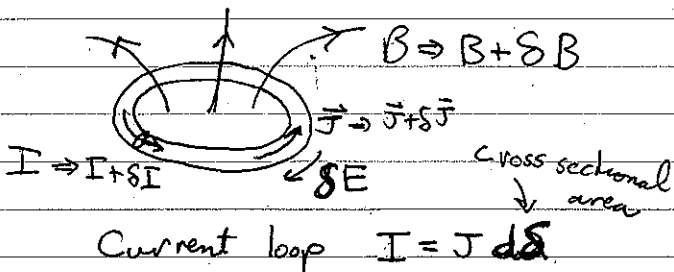
In rest frame of the coil the force that moves charge must be an electric field

\Rightarrow In rest frame of circuit $\oint_C \vec{E} \cdot d\vec{l} = \oint_C \vec{E} \cdot d\vec{l} = -\frac{1}{c} \int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a}$

New \vec{E} -field: $\boxed{\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}}$

Energy in the Magnetic Field

Work done to establish a current distribution



As the current increased the flux of B increases there is a "back EMF" to oppose this change. We must do work to balance this force.

Rate at which work is done: $I \Rightarrow I + \delta I$

$$\frac{d}{dt}(\delta W) = - \int \vec{J} \cdot \delta \vec{E} d^3x = - \int \vec{J} dS \oint \delta \vec{E} \cdot d\vec{l} = \frac{1}{c} \int dS \int \delta \vec{B} \cdot \frac{d\vec{A}}{\delta t}$$

$$\Rightarrow \delta W = \frac{1}{c} \int dS \vec{J} \int \delta \vec{B} \cdot d\vec{A} = \frac{1}{c} \int dS \vec{J} \int \nabla \times (\delta \vec{A}) \cdot d\vec{A}$$

$$= \frac{1}{c} \int dS \vec{J} \oint \delta \vec{A} \cdot d\vec{l} = \frac{1}{c} \int \vec{J} \cdot \delta \vec{A} d^3x$$

$$\therefore W = \frac{1}{2c} \int \vec{J} \cdot \vec{A} d^3x = U_B \text{ (Potential energy stored in } \vec{B} \text{)}$$

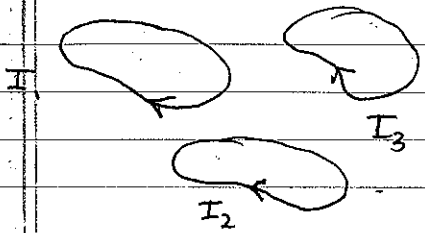
Express in terms of the field: ~~$\frac{1}{2c} \int \vec{J} \cdot \vec{A} d^3x$~~

$$U = \frac{1}{2c} \int \left(\frac{c}{4\pi} \nabla \times \vec{B} \right) \cdot \vec{A} d^3x$$

Integrate by parts

$$U_B = \frac{1}{8\pi} \int \vec{B} \cdot \vec{B} d^3x$$

Collection of loops



$$U_B = \frac{1}{2c} \int d^3x \vec{J} \cdot \vec{A}$$

$$= \sum_i \frac{1}{2c} \oint d\vec{l}_i \vec{I}_i \cdot \vec{A}(\vec{x}_i)$$

$$\vec{A}(\vec{x}) = \frac{1}{c} \int \frac{d^3x' \vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} = \frac{1}{c} \sum_j \oint \frac{d\vec{l}_j \vec{I}_j}{|\vec{x} - \vec{x}_j|}$$

$$\Rightarrow U_B = \frac{1}{2} \sum_{ij} \left(\frac{1}{c^2} \iint \frac{d\vec{l}_i \cdot d\vec{l}_j}{|\vec{x}_i - \vec{x}_j|} \right) I_i I_j$$

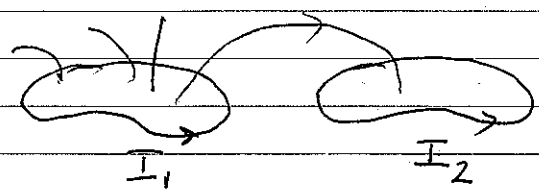
$$\Rightarrow U_B = \frac{1}{2} \sum_{ij} L_{ij} I_i I_j, \quad L_{ij} = \frac{1}{c^2} \iint \frac{d\vec{l}_i \cdot d\vec{l}_j}{|\vec{x}_i - \vec{x}_j|}$$

L_{ij} = coefficient of mutual inductance
 = L_{ji} (reciprocity), $L_{ii} \equiv L$ = self-inductance

Single loop $U_B = \frac{1}{2} L I^2$, L geometrical

Units $[L] = \left[\frac{1}{c^2} \right] \text{cm} \equiv \text{Oersted} \quad (\text{S.I.: Henry})$
 in Gaussian

Calculating inductance
 (look at changing flux)



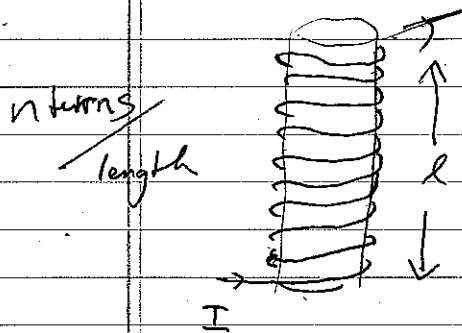
$$\Phi_{2,1} = \int_{S_2} \vec{B}_1 \cdot d\vec{a}_2 = \int_S (\nabla \times \vec{A}_1) \cdot d\vec{a}_2 = \oint \vec{A}_1 \cdot d\vec{l}_2$$

$$= \frac{1}{c} \left(\iint \frac{d\vec{l}_1 \cdot d\vec{l}_2}{|\vec{x}_2 - \vec{x}_1|} \right) I_1 = c L_{21} I_1$$

$$\Rightarrow \left[L_{2,1} = \frac{\Phi_{21}}{c I_1}, \quad L = \frac{\Phi}{c I} \right] \quad \mathcal{E} = -\frac{1}{c} \frac{d\Phi}{dt} \Rightarrow \boxed{\mathcal{E} = -L \frac{dI}{dt}}$$

Inductors oppose changing ~~current~~ current

Example: Solenoid



Assume current I

$$\Rightarrow \text{Uniform } B = \frac{4\pi}{c} n I$$

Total flux:

$$\Phi = N_{\text{turns}} BA = n l B A$$

$$\Phi = \frac{4\pi}{c} n^2 l A I$$

$$\Rightarrow L = \frac{\Phi}{c I} = \frac{4\pi n^2 l A}{c^2}$$