

## Lecture #10: Maxwell's Eqns and Conservation Law

Summary of field eqns up to this point

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J}$$

But we cheated. Ampère's Law here is for steady currents, but we've used it in Faraday's Law. When is it valid? (See below)

Maxwell generalizes Ampère's Law

Add an additional piece:  $\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J} + \vec{J}_d$   
 $\vec{J}_d$  ← Maxwell's Displacement current

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = 0 = \frac{4\pi}{c} (\vec{\nabla} \cdot \vec{J}) + \vec{\nabla} \cdot \vec{J}_d$$

But  $\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$  : Conservation of charge

$$\Rightarrow \vec{\nabla} \cdot \vec{J}_d = \frac{4\pi}{c} \frac{\partial \rho}{\partial t} = \frac{4\pi}{c} \frac{\partial}{\partial t} \left( \frac{1}{4\pi} \vec{\nabla} \cdot \vec{E} \right) = \vec{\nabla} \cdot \left( \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \right)$$

$$\Rightarrow \vec{\nabla} \cdot \left( \vec{J}_d - \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \right) = 0$$

$\therefore$   $\vec{J}_d = \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$  +  ~~$\vec{J}_d$~~  Empirical fact  
Maxwell's Displacement current Where  $\vec{\nabla} \cdot \vec{Q} = 0$

General Form  
of  
Ampère's Law

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$$

Changing  $\vec{E}$  generates  $\vec{B}$

(#10.1)

Why was the displacement current missed empirically?

Compare scales: Suppose  $T$  is characteristic time scale ~~scale~~ of change,  $L$  is distance

Faraday:  $\vec{\nabla} \times \vec{E} = \frac{1}{c} \frac{\partial \vec{B}}{\partial t} \Rightarrow \frac{E}{L} = \frac{1}{cT} B$

Ampere:  $\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \Rightarrow \frac{B}{L} \sim \frac{J}{c} + \left(\frac{L}{cT}\right)^2 \frac{B}{L}$

Conduction current will mask effects of displacement current if

$$\frac{L}{cT} \ll 1 \Rightarrow \boxed{\frac{1}{T} \ll \frac{c}{L}}$$

Fact: Magnetic forces much weaker than electric unless relativistic  $v \sim c$

Quasi-static Ignore displacement current

Valid if  $cT \gg L$

However, even in non-relativistic situation, if

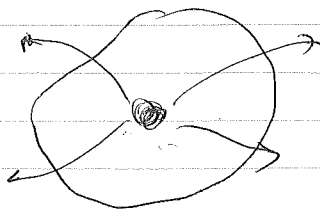
$L \gg cT$  (i.e. far away from circuit)

effects of displacement current are important

$\Rightarrow$  Radiation!

Conservation Laws

General form:  $Q =$  conserved quantity



Define  $\rho_Q =$  density of  $Q$   $\left[ \frac{Q}{L^3} \right]$

$\vec{F}_Q =$  Flux density of  $Q$   $\left[ \frac{Q/T}{L^2} \right]$

$R_Q =$  Rate of creation of  $Q/\text{vol}$   $\left[ \frac{Q}{T L^3} \right]$

Conservation  $\equiv$  detailed balance

$$\frac{dQ}{dt} = -(\text{Rate of flow of } Q \text{ out of volume}) + \left( \begin{array}{c} \text{Rate of creation} \\ \text{of } Q \\ \text{inside volume} \end{array} \right)$$

$$\frac{d}{dt} \int_V \rho_Q d^3x = - \oint_S \vec{F}_Q \cdot d\vec{a} + \int_V R_Q d^3x$$

Differential form

$$\boxed{\frac{\partial \rho_Q}{\partial t} = -\nabla \cdot \vec{F}_Q + R_Q}$$

e.g.: charge  $\frac{\partial \rho}{\partial t} = -\nabla \cdot \vec{J}$  (No creation rate)

Energy in E/M field

$$U = \frac{1}{8\pi} \int (\vec{E} \cdot \vec{E} + \vec{B} \cdot \vec{B}) d^3x, \quad u = \frac{(\vec{E}^2 + \vec{B}^2)}{8\pi} \text{ energy density}$$

$$\frac{\partial u}{\partial t} = \frac{1}{4\pi} (\vec{E} \cdot \frac{\partial \vec{E}}{\partial t} + \vec{B} \cdot \frac{\partial \vec{B}}{\partial t}) = \frac{1}{4\pi} (\vec{E} \cdot (c\vec{\nabla} \times \vec{B} - 4\pi \vec{J}) + \vec{B} \cdot (-c\vec{\nabla} \times \vec{E}))$$

$$= \frac{c}{4\pi} (\vec{E} \cdot (\vec{\nabla} \times \vec{B}) - \vec{B} \cdot (\vec{\nabla} \times \vec{E})) - \vec{J} \cdot \vec{E}$$

$$= -\frac{c}{4\pi} \vec{\nabla} \cdot (\vec{E} \times \vec{B}) - \vec{J} \cdot \vec{E}$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{S} = -\vec{J} \cdot \vec{E}, \quad \vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B} \text{ Poynting vector}}$$

Poynting's theorem

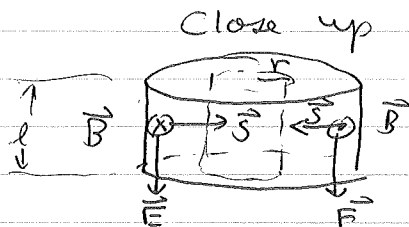
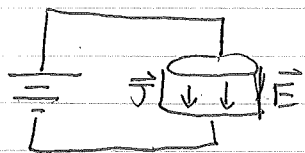
Poynting vector:  $\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B}$  : energy flux density

$\int_{\text{Surface}} \vec{S} \cdot d\vec{a} =$  rate of energy flow through surface

$-\vec{J} \cdot \vec{E} =$  Rate of creation of energy density in field  
 $= -(\text{Ohmic heating rate / volume})$

• Examples of Energy Conservation

Ohmic dissipation: Resistor



$$B = \frac{2 I_{enc}(r)}{cr}$$

$$= \frac{2\pi Jr}{c}$$

(ignore displacement current  $r \ll cT$ )

$$\Rightarrow \vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B} = -\frac{r}{2} J E \hat{e}_\rho$$

Flux through side of ~~side~~ cylinder:  $\int_{\text{side}} \vec{S} \cdot d\vec{a} = - (2\pi r l) \frac{r}{2} J E =$   
↑  
outward

$$\Rightarrow \int \vec{S} \cdot d\vec{a} = - (\pi r^2 l) J E = -IV \quad \text{Ohmic dissipation}$$

Uniqueness of  $\vec{S}$ ? Physics  $\vec{\nabla} \cdot \vec{S}$

In statics more <sup>natural</sup> ~~physical~~ choice  $\vec{S} = \vec{J} \phi$  ( $\frac{IV}{\text{Area}}$ )

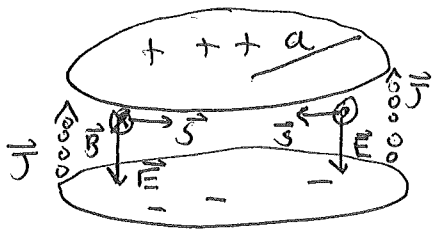
Check:  $\vec{\nabla} \cdot (\phi \vec{J}) = \vec{J} \cdot (\vec{\nabla} \phi) + \phi (\vec{\nabla} \cdot \vec{J})^0 = -\vec{J} \cdot \vec{E}$

$$\Rightarrow \vec{\nabla} \cdot \vec{S} = -\vec{J} \cdot \vec{E} \quad \checkmark$$

Not equivalent in dynamics, including radiation

Example: Create  $E + M$  energy: Charge a capacitor  
by moving current against field

(#10.4)



Suppose current is moved only  
along the edges of the plate  
 $\Rightarrow$  only displacement current inside

Ampere  $\oint \vec{B} \cdot d\vec{l} = \frac{1}{c} \int \frac{\partial \vec{E}}{\partial t} \cdot d\vec{a} \Rightarrow B(r) = \frac{1}{2\pi r c} \int \frac{\partial \vec{E}}{\partial t} \cdot d\vec{a}$

Neglect fringing fields  $\Rightarrow E$  uniform inside  $\Rightarrow \vec{B} = -\frac{r}{2c} \frac{\partial E}{\partial t} \hat{\phi}$

$$\Rightarrow \vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B} = -\frac{r}{8\pi} |\vec{E}| \left| \frac{\partial \vec{E}}{\partial t} \right| \hat{e}_\phi = -\frac{r}{16\pi} \frac{\partial}{\partial t} |\vec{E}|^2 \hat{e}_\phi$$

Just inside the cylinder

$$\int \vec{S} \cdot d\vec{a} = - (2\pi a d) \frac{a}{16\pi} \frac{\partial}{\partial t} |\vec{E}|^2 = - (\pi a^2 d) \frac{\partial}{\partial t} \left( \frac{E^2}{8\pi} \right)$$

distance between plate

What happened to  $\vec{B}$ ? Quasi-static, ignore back EMF

$$\Rightarrow \vec{\nabla}_x \cdot \vec{E} \approx 0 \Rightarrow \vec{E}(t) \approx -4\pi \frac{Q(t)}{A} \vec{e}_z$$

Energy created at edge  $a$  flows into the capacitor

$$\begin{aligned} -\int \vec{J} \cdot \vec{E} d^3x &= \cancel{\dots} = -\int \vec{J} \cdot d\vec{a} dz \vec{E} = \left( \frac{dQ}{dt} \right) |\vec{E}| d \\ &= \left( \frac{A}{4\pi} \frac{\partial |\vec{E}|}{\partial t} \right) |\vec{E}| d = (Ad) \frac{\partial}{\partial t} \frac{|\vec{E}|^2}{8\pi} \quad \checkmark \end{aligned}$$

This is an approximation since energy created at edge  
immediately appears inside capacitor: Neglect radiation.

Valid if  $T \gg L/c$