Lecture #11: Conservation Laws Continued

Conservation of momentum   Maxwell Stress Tensor

Let us first consider electrostatic forces: $\vec{F}_E = \int \rho \vec{E}\, d^3x$

$$\Rightarrow \vec{F}_E = \int d^3x \left(\frac{\vec{\nabla}\cdot\vec{E}}{4\pi}\right)\vec{E} \quad \Rightarrow \quad F_i = \frac{1}{4\pi}\int d^3x\, (\partial_j E_j) E_i$$

$$= \frac{1}{4\pi}\int d^3x \left[\partial_j (E_j E_i) - E_j \partial_j E_i\right]$$

Statics $\vec{\nabla}\times\vec{E} = 0 \Rightarrow \dfrac{\partial E_i}{\partial x_j} = \dfrac{\partial E_j}{\partial x_i}$

$$\Rightarrow F_i = \frac{1}{4\pi}\int d^3x \left[\partial_j (E_j E_i) - \frac{1}{2}\partial_i (E_j E_j)\right]$$

$$= \frac{1}{4\pi}\int d^3x\, \partial_j \left[E_j E_i - \frac{1}{2}\delta_{ij} |\vec{E}|\right]^2$$

Define Maxwell Stress tensor: $\boxed{\overleftrightarrow{T} = \frac{1}{4\pi}\left(\vec{E}\vec{E} - \frac{1}{2}\hat{\mathbb{1}} E^2\right)}$

Symmetric

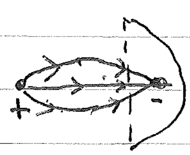
$$\Rightarrow \boxed{\vec{F}_E = \int d^3x \left(\vec{\nabla}\cdot\overleftrightarrow{T}\right) = \oint da\, \hat{n}\cdot\overleftrightarrow{T}}$$

$-\overleftrightarrow{T}\cdot\hat{n}\, da$ = Force transmitted across $\hat{n}\, da$

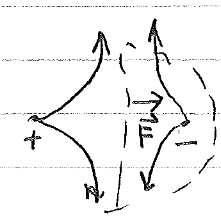
because $\hat{n}$ outward   (not force on $\hat{n}\, da$)

Flux of a vector is a tensor

$\hat{n} \parallel \vec{E} \Rightarrow \overleftrightarrow{T}\cdot\hat{n}$ along $\vec{E} \Rightarrow$ transmits a pull



$\hat{n} \perp \vec{E} \Rightarrow \hat{T}\cdot\hat{n}$ along $-\hat{n} \Rightarrow$ transmits a push



Magnetostatics

, $\vec{\nabla}\cdot\vec{B}=0 \qquad \vec{\nabla}\times\vec{B}=\frac{4\pi}{c}\vec{J}$

$$\Rightarrow \vec{F} = \int\left(\frac{\vec{J}}{c}\times\vec{B}\right)d^3x = \frac{1}{4\pi}\int\left[(\vec{\nabla}\times\vec{B})\times\vec{B}\right]d^3x$$

$$\Rightarrow F_i = \frac{1}{4\pi}\int \epsilon_{ijk}(\vec{\nabla}\times B)_j\, B_k = \frac{1}{4\pi}\int \epsilon_{ijk}\,\epsilon_{jlm}\,\partial_l B_m\, B_k\, d^3x$$

$$= \frac{1}{4\pi}\int(\delta_{il}\delta_{km} - \delta_{im}\delta_{kl})\,(\partial_l B_m B_k)\,d^3x$$

$$= \frac{1}{4\pi}\int\left(\partial_k B_i\, B_k - \frac{1}{2}\partial_i|\vec{B}|^2\right)$$

$$= \frac{1}{4\pi}\int \partial_k\left[B_k B_i - \frac{1}{2}\delta_{ik}\,|\vec{B}|^2\right]d^3x$$

$$\Rightarrow \vec{F} = \int \vec{\nabla}\cdot\overleftrightarrow{T}_B\, d^3x = \oint d\vec{a}\cdot\overleftrightarrow{T}$$

where $\boxed{\overleftrightarrow{T}_B = \frac{1}{4\pi}\left(\vec{B}\vec{B} - \frac{1}{2}\overleftrightarrow{\mathbb{1}}\,|\vec{B}|^2\right)}$

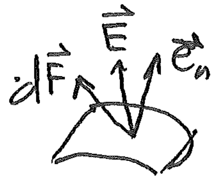
Total Maxwell-Stress-Tensor

$$\boxed{\overleftrightarrow{T} = \frac{1}{4\pi}\left(\vec{E}\vec{E} + \vec{B}\vec{B}\right) - \frac{(E^2+B^2)}{8\pi}\overleftrightarrow{\mathbb{1}}\,)}$$

$\overleftrightarrow{T}$ not so useful in statistics, but important in dynamics

Generally: $d\vec{F} = -da\,\vec{e}_n \cdot \overleftrightarrow{T}$ is the force transmitted by the field across the area normal to $\vec{e}_n$

Consider an arbitrary surface element $d\vec{a} = \vec{e}_n da$ and the local field at that point



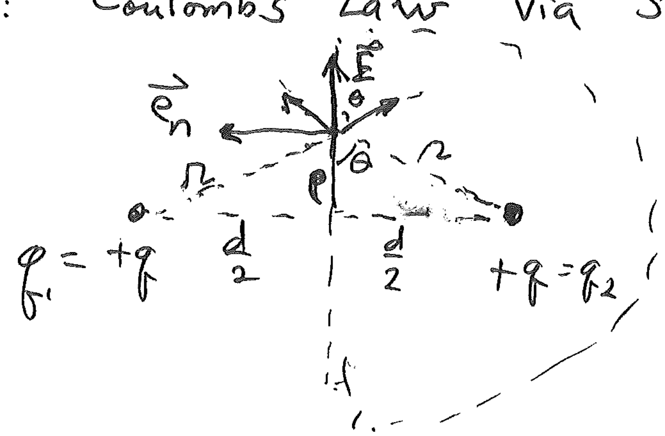
Facts

(i) $d\vec{F}$ is in the plane containing $\vec{E}$ and $\vec{e}_n$

(ii) Angle between $\vec{E}$ and $\vec{e}_n$ = Angle between $d\vec{F}$ and $\vec{E}$

(iii) $|d\vec{F}| = \frac{|\vec{E}|^2}{8\pi} da \Rightarrow \frac{|\vec{E}|^2}{8\pi}$ = electrostatic pressure

Example: Coulomb's Law via Stress Tensor



To find force on q_2 we will calculate flux of $\overleftrightarrow{T}$ through close hemisphere, radius R

$$\Rightarrow \vec{F} = \oint_{\text{hemisphere}} d\vec{a} \cdot \overleftrightarrow{T} = \int_{\text{disk}} d\vec{a} \cdot \overleftrightarrow{T} + \int_{\text{dome}} d\vec{a} \cdot \overleftrightarrow{T}$$

The calculation is easiest with $R \to \infty$

On dome $da \sim \frac{1}{R^2}$ whereas $T \propto |E|^2 \propto \frac{1}{R^4}$

$$\Rightarrow \lim_{R\to\infty} \int_{\text{dome}} d\vec{a} \cdot \overleftrightarrow{T} \to 0$$

$$\Rightarrow \quad \vec{F} = \int_{\text{disk}} d\vec{a} \cdot \overleftrightarrow{T}$$

on disk $\vec{E} \perp \vec{e}_n \quad \Rightarrow \quad d\vec{a} \cdot \overleftrightarrow{T} = -\vec{e}_n \frac{|\vec{E}|^2}{8\pi} da$

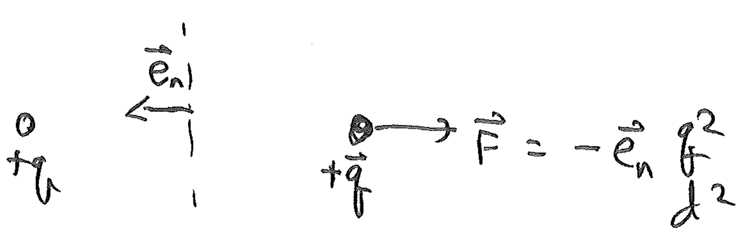
On the disk, $|\vec{E}|$ is azimuthally symmetric, depending only on the cylindrical radius ρ

$$|\vec{E}(\rho)| = 2\frac{q}{r^2}\cos\theta = 2\frac{q\rho}{r^3} = \frac{2q\rho}{(\rho^2 + \frac{d^2}{4})^{3/2}}$$

$$da = 2\pi\rho\, d\rho$$

$$\Rightarrow \vec{F} = -\vec{e}_n \int_0^\infty \rho\, d\rho \frac{|\vec{E}|^2}{4} = -\vec{e}_n q^2 \int_0^\infty \frac{\rho^3}{(\rho^2 + \frac{d^2}{4})^3} d\rho$$

$$= -\vec{e}_n q^2 \left[-\frac{d^2 + 8\rho^2}{16(\rho^2 + \frac{d^2}{4})^2} \right]_0^\infty = -\vec{e}_n \frac{q^2}{d^2}$$



Coulomb's Law !

L#11.4

Conservation of momentum in E&M

Newton's Law of action - reaction

$$\vec{F}\Big|_{\text{fields on sources}} = -\vec{F}\Big|_{\text{sources on fields}}$$

$$\Rightarrow \frac{d\vec{P}}{dt}\Big|_{\text{sources}} = -\frac{d\vec{P}}{dt}\Big|_{\text{fields}}$$

$$\Rightarrow \frac{d\vec{P}_{\text{fields}}}{dt} = -\int d^3x \left(\rho(\vec{x},t)\,\vec{E}(\vec{x},t) + \frac{\vec{J}}{c}(\vec{x},t)\times\vec{B}(\vec{x},t)\right)$$

Thus, we interpret

$$\boxed{\begin{aligned}\vec{R}_{\vec{p}} &= -\left(\rho(\vec{x},t)\,\vec{E}(\vec{x},t) + \frac{\vec{J}(\vec{x},t)}{c}\times\vec{B}(\vec{x},t)\right)\\ &= \frac{\text{Rate of creating momentum in field}}{\text{Volume}}\end{aligned}}$$

Consider now $\vec{\nabla}\cdot(-\overleftrightarrow{T}) =$ local flow of momentum in field

$$-\overleftrightarrow{T} = \left(\frac{E^2+B^2}{8\pi}\right)\mathbb{1} - \frac{1}{4\pi}\left(\vec{E}\,\vec{E} + \vec{B}\,\vec{B}\right)$$

$$\left(\vec{\nabla}\cdot(-\overleftrightarrow{T})\right)_i = \partial_j(-T_{ji})$$

$$= \partial_i\left(\frac{E^2+B^2}{8\pi}\right) - \frac{1}{4\pi}\left[\partial_j(E_jE_i) + \partial_j(B_jB_i)\right]$$

L#11.5

Asides:

$$\partial_i\left(\frac{E^2+B^2}{8\pi}\right) = \frac{1}{8\pi}\partial_i\left(E_jE_j + B_jB_j\right) = \frac{1}{4\pi}\left(E_j\partial_iE_j + B_j\partial_iB_j\right)$$

$$\partial_j\left(E_jE_i + B_jB_i\right) = E_i(\vec{\nabla}\cdot\vec{E}) + E_j\partial_jE_i + B_i(\vec{\nabla}\cdot\vec{B}) + B_j\partial_jB_i$$

$$\therefore \left(\vec{\nabla}\cdot(-\overleftrightarrow{T})\right)_i = \frac{1}{4\pi}\Big(-E_i(\vec{\nabla}\cdot\vec{E}) - B_i(\vec{\nabla}\cdot\vec{B}) + E_j(\partial_iE_j - \partial_jE_i) + B_j(\partial_iB_j - \partial_jB_i)\Big)$$

Aside:

$$E_j(\partial_iE_j - \partial_jE_i) = E_j\epsilon_{ijk}(\vec{\nabla}\times\vec{E})_k = -\left(\vec{E}\times(\vec{\nabla}\times\vec{E})\right)_i$$

and similarly for $\vec{B}$

$$\therefore \vec{\nabla}\cdot(-\overleftrightarrow{T}) = \frac{1}{4\pi}\Big(-\vec{E}(\vec{\nabla}\cdot\vec{E}) - \vec{B}(\vec{\nabla}\cdot\vec{B}) + \vec{E}\times(\vec{\nabla}\times\vec{E}) + \vec{B}\times(\vec{\nabla}\times\vec{B})\Big)$$

Now use Maxwell's Equations:

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \qquad \vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

$$\Rightarrow \vec{\nabla}\cdot(-\overleftrightarrow{T}) = -\rho\vec{E} + \vec{E}\times\left(\frac{-1}{4\pi c}\frac{\partial\vec{B}}{\partial t}\right)$$

$$+ \vec{B}\times\frac{\vec{J}}{c} + \vec{B}\times\left(\frac{1}{4\pi c}\frac{\partial\vec{E}}{\partial t}\right)$$

$$\vec{\nabla}\cdot(-\overleftrightarrow{T}) = -\rho\vec{E} - \frac{\vec{J}}{c}\times\vec{E} - \frac{1}{4\pi c}\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$

$\Rightarrow$ Conservation of momentum

$$\boxed{\frac{\partial}{\partial t}\vec{g} + \vec{\nabla}\cdot(-\overleftrightarrow{T}) = \vec{R}_p}$$

$\vec{g} \equiv \frac{1}{4\pi c}\vec{E}\times\vec{B} = \frac{\vec{S}}{c^2}$ = momentum density in field

$-\overleftrightarrow{T} = \left(\frac{E^2+B^2}{8\pi}\right)\overleftrightarrow{\mathbb{1}} - \frac{1}{4\pi}(\vec{E}\vec{E} + \vec{B}\vec{B})$

= momentum flux density of field

$\vec{R}_p = -(\rho\vec{E} + \frac{\vec{J}}{c}\times\vec{B})$ = rate of creation of momentum in field

L#11.7

Angular Momentum Conservation

let $\vec{L}$ = angular moment of source

$$\frac{d\vec{L}_{source}}{dt} = \int d^3x \; \vec{x} \times \vec{f}(\vec{x}) = \int d^3x \; \vec{x} \times \left(\rho(\vec{x},t)\,\vec{E}(\vec{x},t) + \frac{\vec{J}(\vec{x},t)}{c} \times \vec{B}(\vec{x},t) \right)$$

$$= -\frac{d\vec{L}_{field}}{dt}$$

$$\Rightarrow -\vec{x} \times \vec{f}(\vec{x}) = -\vec{x} \times \left(\rho\vec{E} + \frac{\vec{J}}{c} \times \vec{B}\right) = \vec{x} \times \vec{R}_p$$

$\equiv \vec{R}_{\vec{L}}$ = rate of creation of angular momentum in field volume

$$\Rightarrow \boxed{\frac{\partial}{\partial t}(\vec{x} \times \vec{g}) + \vec{\nabla} \cdot (-\vec{x} \times \overleftrightarrow{T}) = \vec{R}_L}$$

Angular momentum density in field: $\vec{\mathcal{L}} \equiv \vec{x} \times \vec{g} = \vec{x} \times \left(\frac{\vec{E} \times \vec{B}}{4\pi c}\right)$

Flux density of anglar momentum:

$-\vec{x} \times \overleftrightarrow{T}$ = a big mess!