Lecture #12: Plane wave solutions in free space

Maxwell eqns in free space $\rho=0$ $\vec{J}=0$

$$\vec{\nabla}\cdot\vec{E}=0 \qquad \vec{\nabla}\cdot\vec{B}=0$$

$$\vec{\nabla}\times\vec{E}=-\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B}=\frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

Coupled PDE's. Changing $\vec{B}$ generates, changing $\vec{E}$ which in turn generates changing $\vec{B}$ ...

$\Longrightarrow$ Wave Propagation

Decouple $\Rightarrow$ Take second derivative:

$$\vec{\nabla}\times\vec{\nabla}\times\vec{E}=-\frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B}) \Rightarrow \vec{\nabla}(\cancelto{0}{\vec{\nabla}\cdot\vec{E}})-\nabla^2\vec{E}=-\frac{1}{c}\frac{\partial^2\vec{E}}{\partial t}$$

$$\Longrightarrow \boxed{\begin{aligned}\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E}&=0\\ \left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{B}&=0\end{aligned}}$$

Vector wave eqn. in 3D

Propagation speed $\underline{c}$

Each Cartesian component of $\vec{E}$ and $\vec{B}$ satisfy 3D wave eqn.

But $\vec{E},\vec{B}$ are constrained by Maxwells eqns

$\vec{\nabla}\cdot\vec{E}=0$, $\vec{\nabla}\cdot\vec{B}=0$ $\Longrightarrow$ Two components of $\vec{E},\vec{B}$ determine third

$\vec{\nabla}\times\vec{E}=-\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$, $\vec{\nabla}\times\vec{B}=\frac{1}{c}\frac{\partial\vec{E}}{\partial t}$ $\Rightarrow$ Fixing $\vec{E}$ determines $\vec{B}$ and vice versa

Monochromatic solutions and complex notation

Most general form $\vec{E}(\vec{x},t) = \mathrm{Re}\left(\underset{\sim}{\vec{E}}(\vec{x})\, e^{-i\omega t}\right)$

$$\underset{\sim}{E}(\vec{x}) = \underset{\substack{\uparrow \\ \text{take real for now} \\ \text{(polarization)}}}{\hat{e}}\ \underbrace{e^{i\phi(\vec{x})}}_{\text{phase}}\ \underbrace{E_0(\vec{x})}_{\text{Amplitude}} \quad : \quad \text{"Complex Amplitude"}$$

$$\Rightarrow \vec{E}(\vec{x},t) = \hat{e}\ E_0(\vec{x})\ \cos(\phi(x) - \omega t)$$

$$= \hat{e}\left[A(\vec{x})\cos\omega t + B(\vec{x})\sin\omega t\right]$$

} General solution for field oscillating like ωt

$$\frac{\partial}{\partial t}\left(\underset{\sim}{E}(\vec{x})e^{-i\omega t}\right) = -i\omega t\ \underset{\sim}{E}e^{-i\omega t}$$

$\Rightarrow$ In free space, Maxwell's Eqns assume monochromatic field

$$\boxed{\vec{\nabla}\cdot\underset{\sim}{E} = 0, \quad \vec{\nabla}\cdot\underset{\sim}{B} = 0, \quad \nabla\times\underset{\sim}{E} = \frac{i\omega}{c}\underset{\sim}{B}, \quad \nabla\times\underset{\sim}{B} = -\frac{i\omega}{c}\underset{\sim}{E}}$$

I will drop $\sim$ and assume we take real part in end

$$-\vec{\nabla}\times\vec{\nabla}\times\vec{E} = \nabla^2\vec{E} = -i\frac{\omega}{c}\vec{\nabla}\times\vec{B} = -\frac{\omega^2}{c^2}\vec{E}$$

$$\Rightarrow \boxed{\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\vec{E} = 0, \quad \left(\nabla^2 + \frac{\omega^2}{c^2}\right)\vec{B} = 0}$$

Helmholtz equation: $(\nabla^2 + a^2)\psi = 0$

Constraint: $\boxed{\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{B} = -i\frac{c}{\omega}\vec{\nabla}\times\vec{E}}$

Plane wave solution: Ansatz $\vec{E} = \vec{E}_0 \, e^{i\vec{k}\cdot\vec{x}}$

vector amplitude

$$\vec{\nabla} e^{i\vec{k}\cdot\vec{x}} = i\vec{k}\, e^{i\vec{k}\cdot\vec{x}} \quad \left(\text{Proof } \frac{\partial}{\partial y} e^{i\vec{k}\cdot\vec{x}} = i k_y \, e^{i\vec{k}\cdot\vec{x}}\right)$$

$$\Rightarrow \nabla^2 e^{i\vec{k}\cdot\vec{x}} = -|\vec{k}|^2 e^{i\vec{k}\cdot\vec{x}}$$

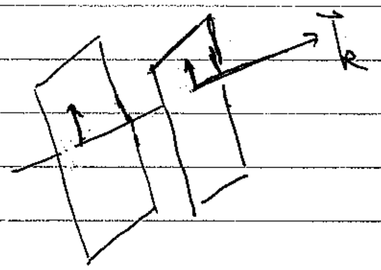
$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\vec{E} \Rightarrow \left(-|\vec{k}|^2 + \frac{\omega^2}{c^2}\right)\vec{E}_0 e^{i\vec{k}\cdot\vec{x}} = 0$$

Solution if $\boxed{|\vec{k}| = \frac{\omega}{c}}$ Wave number

Dispersion relation

Real field

$$\vec{E}(\vec{x},t) = \vec{E}_0 \cos(\vec{k}\cdot\vec{x} - \omega t + \phi_0)$$



At a fixed time, locus of points $\perp$ to $\vec{k}$ have same $\vec{E}$

$\Rightarrow$ $\boxed{\text{Wave fronts are planes } \perp \text{ to } \vec{k}}$

$\vec{k}$ = wave vector $\rightarrow$ direction of propagation

Period in space $|k|\lambda = 2\pi \Rightarrow \lambda = \frac{2\pi}{k}$ wave length

Period in time $\omega T = 2\pi \Rightarrow T = \frac{2\pi}{\omega} \quad \nu = \frac{1}{T}\left(\text{cycles}/\text{sec}\right)$

Phase velocity $\omega/k = c$

Constraint: $\vec{\nabla}\cdot\vec{E} = 0 \quad \vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \vec{k}\cdot\vec{E} = \vec{k}\cdot\vec{B} = 0$

$\Rightarrow$ $\boxed{\vec{E} \text{ and } \vec{B} \perp \text{ to } \vec{k} \Rightarrow \text{transverse wave}}$

$\vec{\nabla}\times\vec{E} = i\frac{\omega}{c}\vec{B} = ik\vec{B} \Rightarrow \vec{B} = \hat{k}\times\vec{E}$ $\longrightarrow$ $|\vec{B}| = |\vec{E}|$, $\vec{B} \perp \vec{E}$

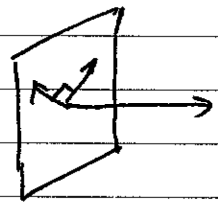
$\longrightarrow$ $\vec{E}\times\vec{B}$ direction $\vec{k}$

Polarization

Given a plane wave propagating in $\hat{k}$ direction, the most general field:

$$\vec{E}(\vec{x},t) = E_1 \cos(\vec{k}\cdot\vec{x}-\omega t+\phi_1)\hat{e}_1 + E_2 \cos(\vec{k}\cdot\vec{x}-\omega t+\phi_2)\hat{e}_2$$

$\hat{e}_1$ and $\hat{e}_2$ are orthonormal unit vectors in plane $\perp$ to $\hat{k}$



$$\hat{e}_1 \times \hat{e}_2 = \hat{k}$$

Complex Notation: $\vec{E}(\vec{x},t) = Re\left\{(a_1 e^{i\phi_1}\hat{e}_1 + a_2 e^{i\phi_2}\hat{e}_2)\, E_0\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}\right\}$

where $E_0 = \sqrt{E_1^2+E_2^2}$ $\quad a_1 = E_1/E_0$, $\quad a_2 = E_2/E_0$ $\quad a_1^2+a_2^2 = 1$

Define complex polarization vector

$$\hat{\epsilon} = \alpha_1 \hat{e}_1 + \alpha_2 \hat{e}_2 \qquad \alpha_1, \alpha_2 \text{ complex numbers}: |\alpha|^2 \leq 1$$

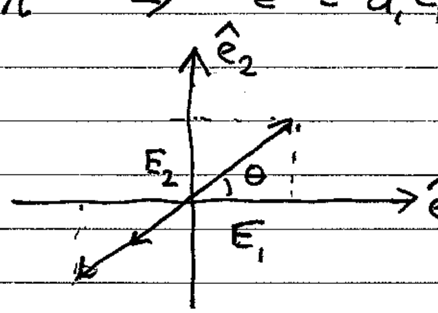
Inner product as in quantum mechanics: $|\hat{\epsilon}|^2 = \hat{\epsilon}^* \cdot \hat{\epsilon}$

unit vector $|\hat{\epsilon}| = |\alpha_1|^2 + |\alpha_2|^2 = 1$

Characterizing polarization state: Only relative phase between $\hat{e}_1$ and $\hat{e}_2$ matter: $\hat{\epsilon} = a_1 \hat{e}_1 + a_2 e^{i\Delta\phi}\hat{e}_2$

- Linear polarization: $\Delta\phi = m\pi \Rightarrow \hat{\epsilon} = a_1\hat{e}_1 \pm a_2\hat{e}_2$

In fixed plane $\vec{E}$ vector oscillates along a line



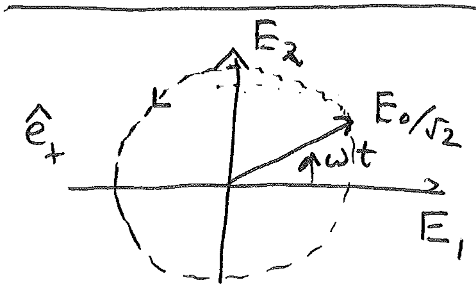
$$\theta = \pm\tan^{-1}\left(\frac{a_2}{a_1}\right)$$

- Circular polarization: $a_1 = a_2 = \frac{1}{\sqrt{2}}$, $\Delta\phi = \pm i$

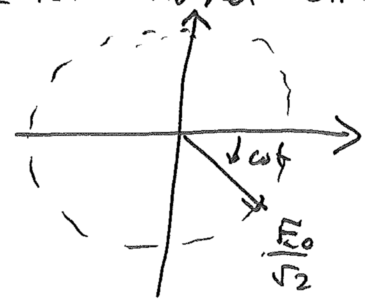
$$\vec{\epsilon}_\pm \equiv (\hat{e}_1 \pm i \vec{e}_2) \frac{1}{\sqrt{2}}$$

at $\vec{x} = 0$ $\vec{E}(\vec{x},t) = \frac{E_0}{\sqrt{2}} (\cos\omega t \, \vec{e}_1 \pm \sin\omega t \, \vec{e}_2)$

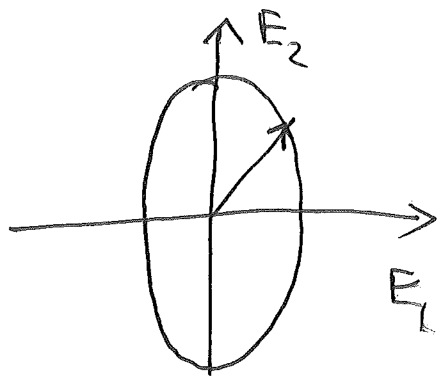
$\hat{e}_+$: Positive helicity ("right-hand" circular)



$\hat{e}_-$: Negative helicity ("left-hand" circular)



- If $a_1 \neq a_2$, $\Delta\phi = \pm i$ $\Rightarrow$ elliptical with major/minor axes on E_1/E_2 Axes



$$\vec{\epsilon} = a_1 \vec{e}_1 + i a_2 \vec{e}_2$$

with $a_2 > a_1$

$$\sqrt{a_1^2 + a_2^2} = 1$$

- Generall polarization: $\Delta\phi \neq n\frac{\pi}{2}$, $n = 0, \pm 1, \pm 2$

$\Rightarrow$ elliptical with major/minor axes tilted

General elliptical

@ origin $\quad \vec{E}(t) = E_{10}\cos\omega t \, \vec{e}_1 + E_{20}\cos(\omega t - \Delta\phi)\, \vec{e}_2$

E_1 and E_2 components as functions of time

$$E_1(t) = E_{10}\cos\omega t \Rightarrow \cos\omega t = \frac{E_1(t)}{E_{10}}$$

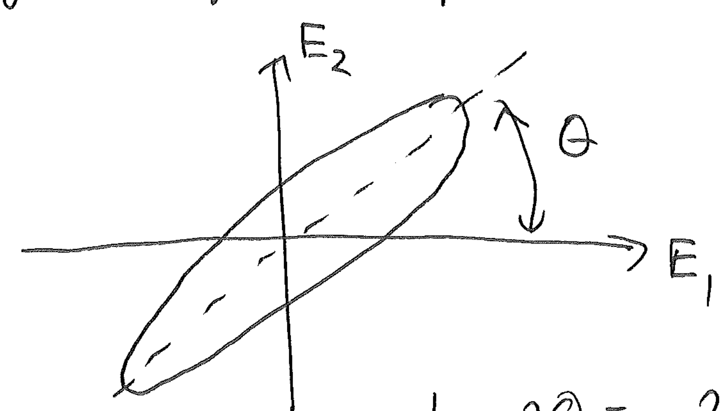
$$E_2(t) = E_{20}\cos(\omega t + \Delta\phi)$$
$$= E_{20}\cos(\omega t)\cos(\Delta\phi) + E_{20}\sin(\omega t)\sin(\Delta\phi)$$

$$\Rightarrow \frac{E_2(t)}{E_{20}} = \cos(\Delta\phi)\cos\omega t + \sin(\Delta\phi)\sin\omega t$$

$$\Rightarrow \frac{E_2(t)}{E_{20}} - \cos(\Delta\phi)\frac{E_1(t)}{E_0} = +\sqrt{1 - \left(\frac{E_1(t)}{E_0}\right)^2}\,\sin(\Delta\phi)$$

$$\Rightarrow \boxed{\left(\frac{E_1}{E_{10}}\right)^2 + \left(\frac{E_2}{E_{20}}\right)^2 - 2\left(\frac{E_1 E_2}{E_{10}E_{20}}\right)\cos\Delta\phi = \sin^2\Delta\phi}$$

equation of an ellipse



$$\tan 2\theta = \frac{2E_{10}E_{20}}{(E_{10}^2 - E_{20}^2)}\cos(\Delta\phi)$$

Stokes vector and Poincaré Sphere

Let us write a general (normalized) polarization vector

$$\vec{\epsilon} = \underset{\uparrow}{\alpha \vec{e}_+} + \beta \vec{e}_- \qquad \vec{e}_\pm = \frac{\vec{e}_x \pm i\vec{e}_y}{\sqrt{2}}$$

positive helicity   negative helicity

Since the relative phase between α & β is all that matters in specify the "type" of polarization, we characterize $\vec{\epsilon}$ by,

$$\vec{\epsilon} = |\alpha|\, \vec{e}_+ + e^{i\phi} |\beta|\, \vec{e}_- ,$$

and since $\vec{\epsilon}^* \cdot \vec{\epsilon} = |\alpha|^2 + |\beta|^2 = 1$, there are only two real numbers that characterize the polarization.

Define $|\alpha| = \cos\frac{\theta}{2}$, $\quad |\beta| = \sqrt{1-|\alpha|^2} = \sin\frac{\theta}{2}$

$$\Rightarrow \boxed{\vec{\epsilon} \Longleftrightarrow (\theta, \phi) : \text{Point on a sphere} \equiv \text{Poincaré sphere}}$$

$\begin{cases} \theta = 0 \\ \alpha = 1 \end{cases} \Rightarrow \vec{\epsilon} = \vec{e}_+$: positive helicity RHC

$\begin{cases} \theta = \pi \\ \beta = 1 \end{cases} \Rightarrow \vec{\epsilon} = \cancel{e^{i\phi}} \vec{e}_-$: negative helicity LHC

$$\underbrace{\theta = \frac{\pi}{2} \Rightarrow |\alpha| = \frac{1}{\sqrt{2}} = |\beta|} \Rightarrow \vec{\epsilon} = \frac{1}{\sqrt{2}}\left(\vec{e}_+ + e^{i\phi}\vec{e}_-\right)$$

$$\Rightarrow \vec{\epsilon} = \left(\frac{1+e^{i\phi}}{2}\right)\vec{e}_x - i\left(\frac{1-e^{i\phi}}{2}\right)\vec{e}_y$$

$$= e^{i\phi/2}\underbrace{\left(\cos\frac{\phi}{2}\,\vec{e}_x + \sin\frac{\phi}{2}\,\vec{e}_y\right)}$$

neglecte Linear polarization

$\left(\theta = \frac{\pi}{2}, \phi = 0\right)$: $\vec{\epsilon} = \vec{e}_x$

$\left(\theta = \frac{\pi}{2}, \phi = \frac{\pi}{2}\right)$: $\vec{\epsilon} = \frac{1}{\sqrt{2}}(\vec{e}_x + \vec{e}_y)$ (45° linear)

$\left(\theta = \frac{\pi}{2}, \phi = \pi\right)$: $\vec{\epsilon} = \vec{e}_y$

$\left(\theta = \frac{\pi}{2}, \phi = \frac{3\pi}{2}\right)$: $\vec{\epsilon} = \frac{1}{\sqrt{2}}(\vec{e}_x - \vec{e}_y)$ (−45° linear)

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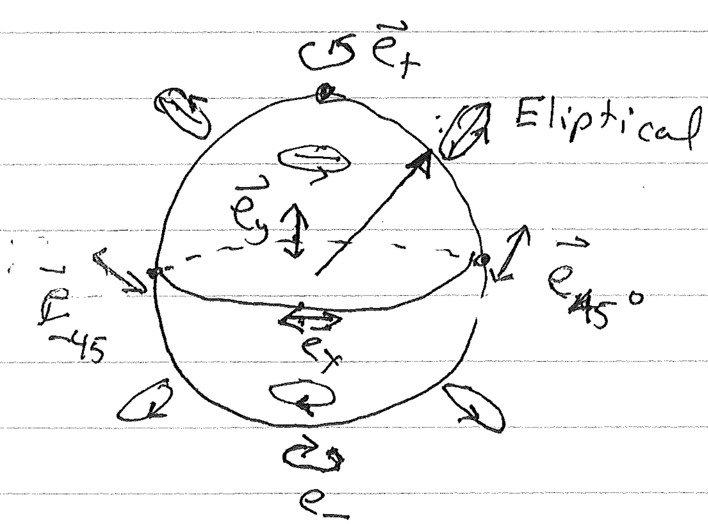
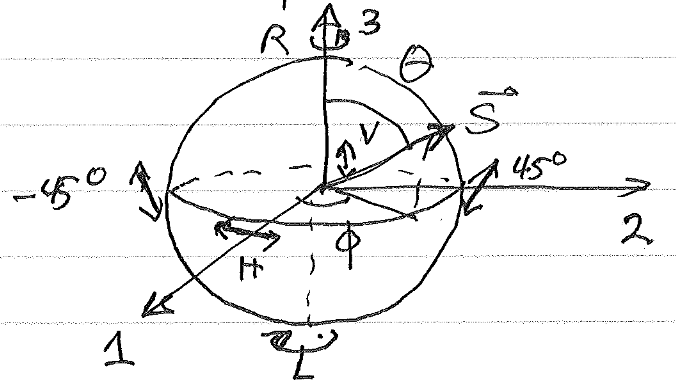


Poincaré sphere

Stokes vector

The vector that specifies the point on the Poincaré sphere is known as the Stokes vector



$$\boxed{S_1 = \sin\theta\cos\phi, \quad S_2 = \sin\theta\sin\phi, \quad S_3 = \cos\theta}$$

Recall with $\vec{E} = |\alpha_+|\vec{e}_+ + e^{i\phi}|\alpha_-|\vec{e}_-$

$$|\alpha_+| = \cos\frac{\theta}{2}, \quad |\alpha_-| = \sin\frac{\theta}{2}$$

$$\alpha_+^*\alpha_- = \sin\frac{\theta}{2}\cos\frac{\theta}{2}\, e^{i\phi} = \frac{1}{2}\sin\theta\, e^{i\phi}$$

$$|\alpha_+|^2 - |\alpha_-|^2 = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \cos\theta$$

$$\therefore \boxed{\begin{aligned} S_1 &= 2\,\mathrm{Re}(\alpha_+^*\alpha_-) = \alpha_+^*\alpha_- + \alpha_-^*\alpha_+ \\ S_2 &= 2\,\mathrm{Im}(\alpha_+^*\alpha_-) = \frac{\alpha_+^*\alpha_- - \alpha_-^*\alpha_+}{i} \\ S_3 &= |\alpha_+|^2 - |\alpha_-|^2 \end{aligned}}$$

Note: The Stokes vector and Poincaré sphere are isomorphic to the Blochvector and Bloch Sphere for a two-level quantum system (e.g. spin 1/2)

Isomorphisms

basis up $|\uparrow\rangle \Longleftrightarrow \vec{e}_+$

basis down $|\downarrow\rangle \Longleftrightarrow \vec{e}_-$

arbitrary $|\psi\rangle = c_\uparrow|\uparrow\rangle + c_\downarrow|\uparrow\rangle \Longleftrightarrow \vec{e} = \alpha_+\vec{e}_+ + \alpha_-\vec{e}_-$

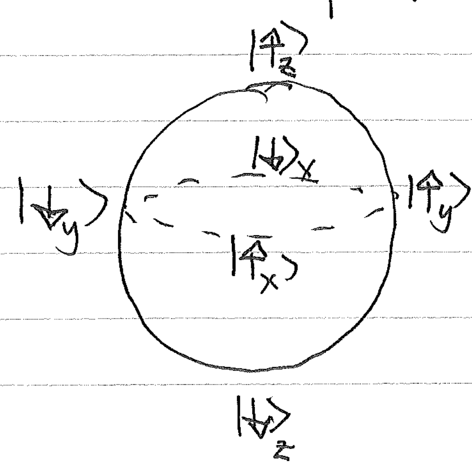
Bloch vector $\vec{R} \Longleftrightarrow$ Stokes vector $\vec{S}$

$$\vec{R} = \langle\psi|\hat{\vec{\sigma}}|\psi\rangle$$ Pauli operators

$$R_3 = |c_\uparrow|^2 - |c_\downarrow|^2$$

$$R_1 = 2\,\mathrm{Re}(c_\uparrow^* c_\downarrow)$$

$$R_2 = 2\,\mathrm{Im}(c_\uparrow^* c_\downarrow)$$



Bloch sphere