Lecture 13 Wave propagation in free Space II

Energy in plane waves

Energy density: $u = \dfrac{|\vec{E}(\vec{x},t)|^2 + |\vec{B}(\vec{x},t)|^2}{8\pi}$

Poynting vector: $\vec{S} = c\,\dfrac{\vec{E}(\vec{x},t) \times \vec{B}(\vec{x},t)}{4\pi}$

For ~~plan~~ monochromatic field oscillating like ω, typically only time averaged effects over a period ~~matter~~ oscillation

Aside: Given two functions oscillating with same frequency ω

$$f(t) = f_0 \cos(\omega t + \phi_1), \quad g(t) = g_0 \cos(\omega t + \phi_2)$$

$$= (\tilde{f}_0\, e^{-i\omega t}) \qquad = Re(\tilde{g}_0\, e^{-\omega t})$$

Complex amplitude $\tilde{f}_0 = f_0\, e^{-i\phi_1}$ $\qquad g_0 = g_0\, e^{-i\phi_2}$

Time average: $\boxed{\langle f(t)\, g(t)\rangle = \int_0^T f(t)\, g(t)\, dt = \frac{1}{2} Re(\tilde{f}_0\, \tilde{g}_0^*) = \frac{1}{2} Re(\tilde{f}_0\, \tilde{g}_0^*)}$

Proof left to class

Plane wave $\vec{E}(\vec{x},t) = Re\{ \underset{\sim}{E}(\vec{x})\, e^{-i\omega t}\} = Re\{ \underset{\sim}{E}_0\, e^{ik\cdot\vec{x} - \omega t}\}$

$\vec{B}(\vec{x},t) = Re\{\underset{\sim}{B}(\vec{x})\, e^{-i\omega t}\}$ $\qquad \underset{\sim}{B}_0 = \hat{k} \times \underset{\sim}{E}_0$

$$\Rightarrow \langle u \rangle = \frac{1}{2} Re\left(\frac{\underset{\sim}{E}_0 \cdot \underset{\sim}{E}_0^*}{8\pi} + \frac{\underset{\sim}{B}_0 \cdot \underset{\sim}{B}_0^*}{8\pi} \right) = \frac{|\underset{\sim}{E}_0|^2}{8\pi} \quad \text{Uniform}$$

Intensity $\langle \vec{S} \rangle = \frac{1}{2} Re\left(\frac{c}{4\pi}\, \underset{\sim}{E}_0 \times \underset{\sim}{B}_0^* \right) = \frac{c}{8\pi} |\underset{\sim}{E}_0|^2\, \hat{k} = \frac{c}{8\pi} |\underset{\sim}{E}_0|^2$

$= c\, u\, \hat{k}$ (uniform ~~as~~ flux density)

Note: Total energy in a plane wave is infinite
Unphysical $\Rightarrow$ Must consider wave packet

Review of Fourier Transform

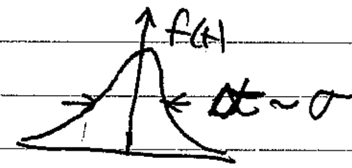
Scalar function $f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \tilde{f}(\omega)\, e^{-i\omega t}$

$\tilde{f}(\omega) = \int_{-\infty}^{\infty} dt\, f(t)\, e^{+i\omega t}$: Fourier transform
Project out component oscillating as $e^{-i\omega t}$

Orthogonal function expansion $\int \frac{dt}{2\pi}\, e^{\pm i(\omega-\omega')t} = \delta(\omega-\omega')$

Completeness $\int \frac{d\omega}{2\pi}\, e^{\pm i\omega(t-t')} = \delta(t-t')$

Example: $f(t) = A e^{-t^2/2\sigma^2}$ Gaussian



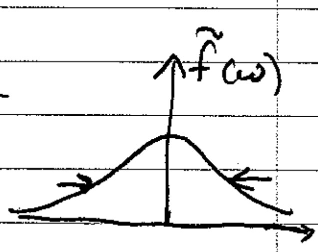
$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} dt\, f(t) e^{i\omega t} = A \int_{-\infty}^{\infty} dt\, \exp\left\{ i\omega t - \frac{t^2}{2\sigma^2} \right\}$$

Complete the square: $-\frac{t^2}{2\sigma^2} + i\omega t = \frac{-(t^2 - 2i\sigma^2\omega t)}{2\sigma^2}$

$$= -\frac{(t - i\sigma^2\omega)^2}{2\sigma^2} - \frac{\sigma^2\omega^2}{2}$$

$$\Rightarrow \tilde{f}(\omega) = A e^{-\frac{\sigma^2\omega^2}{2}} \underbrace{\int_{-\infty}^{\infty} dt\, e^{-(t-i\sigma^2\omega)^2/2\sigma^2}}_{\sqrt{2\pi\sigma^2}}$$

Gaussian



$\Delta\omega \sim \frac{1}{\sigma} = \frac{1}{\Delta t}$

Fourier Uncertainty Principle $\Delta\omega\Delta t \sim 1$

General features of the Fourier ~~#~~ transform

- If $f(t)$ is real

$$\tilde{f}^*(\omega) = \left[\int dt\, f(t)\, e^{+i\omega t}\right]^* = \tilde{f}(-\omega)$$

$\Rightarrow$ $\mathrm{Re}[\tilde{f}(\omega)] = \mathrm{Re}[\tilde{f}(-\omega)]$ (symmetric function)

$\mathrm{Im}[\tilde{f}(\omega)] = -\mathrm{Im}[\tilde{f}(-\omega)]$ (antisymmetric)

- Convolution theorem

$$\tilde{\mathcal{F}}[f(t)\, g(t)] = \int \frac{d\omega'}{2\pi}\, \tilde{f}(\omega-\omega')\, \tilde{g}(\omega')$$

$$\tilde{\mathcal{F}}^{-1}[\tilde{f}(\omega)\, \tilde{g}(\omega)] = \int dt'\, f(t-t')\, g(t')$$

- Translation $\Longleftrightarrow$ phase shift

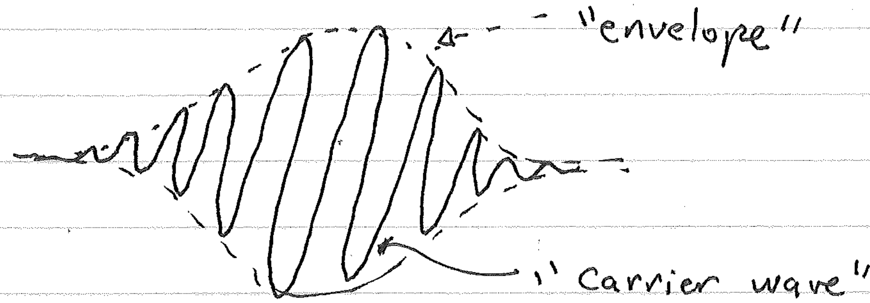
$$\tilde{\mathcal{F}}[f(t-t_0)] = \tilde{f}(\omega)\, e^{i\omega t_0}$$

$$\tilde{\mathcal{F}}^{-1}[\tilde{f}(\omega-\omega_0)] = f(t)\, e^{-i\omega_0 t}$$

- Parseval's theorem

$$\int_{-\infty}^{\infty} dt\, f^*(t)\, g(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \tilde{f}^*(\omega)\, \tilde{g}(\omega)$$ (Inner product of two functions)

Example: Quasimonochromatic Wavepacket (Pulse)



$$F(t) = \underbrace{A e^{-\frac{t^2}{2\Delta t^2}}}_{f(t) \leftarrow \text{envelope}} \cos\omega_0 t \leftarrow \text{carrier}$$

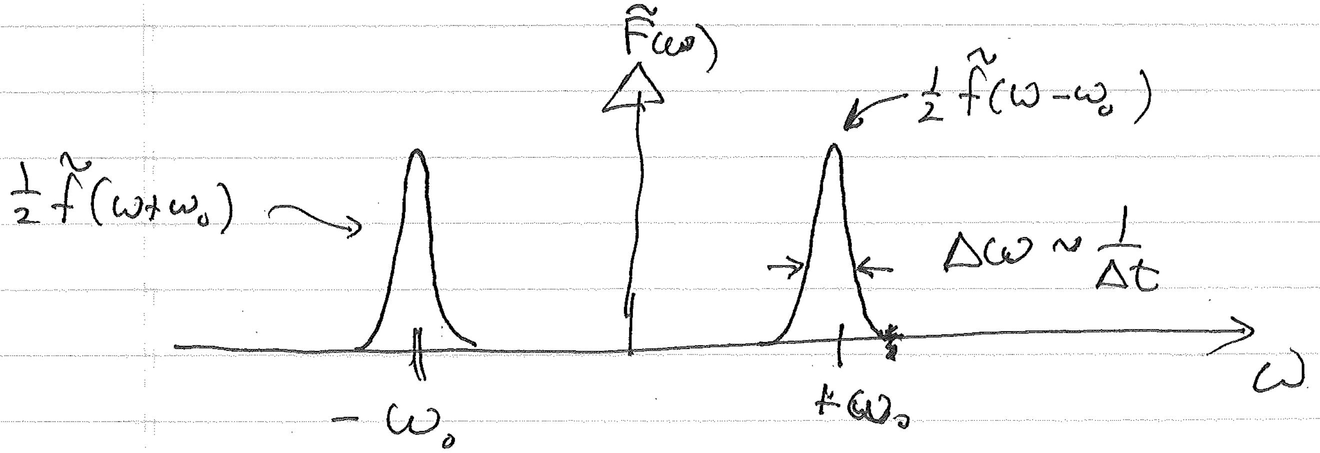
$$\Rightarrow F(t) = \frac{1}{2} f(t) e^{-i\omega_0 t} + \frac{1}{2} f(t) e^{+i\omega_0 t}$$

Quasimonochromatic when $\Delta t \gg \frac{2\pi}{\omega_0} = T$

Fourier transform

$$\tilde{F}(\omega) = \frac{1}{2}\left(\tilde{\mathcal{F}}\left[f(t) e^{-i\omega_0 t}\right] + \tilde{\mathcal{F}}\left[f(t) e^{+i\omega_0 t}\right]\right)$$

$$\Rightarrow \tilde{F}(\omega) = \frac{1}{2}\tilde{f}(\omega - \omega_0) + \frac{1}{2}\tilde{f}(\omega + \omega_0)$$



L#13.5

Application to $\vec{E}, \vec{B}$ waves

- Periodic in both space and time
- Vector wave
- Transverse in free space
- Dispersion relation $\omega^2 + c^2 |\vec{k}|^2 = 0$

General wave in free space

$$\vec{E}(\vec{x},t) = \sum_{\mu} \int \frac{d\omega\, d^3\vec{k}}{(2\pi)^4} \tilde{E}_{\mu}(\vec{k},\omega)\, \vec{e}_{\mu}(\hat{k})\, e^{i(\vec{k}\cdot\vec{x} - \omega t)}\, \delta(\omega^2 - c^2|\vec{k}|^2)$$

$$\vec{k} \cdot \vec{\epsilon}_{\mu}(\hat{k}) = 0$$

$$\vec{B}(\vec{x},t) = \sum_{\mu} \int \frac{d\omega\, d^3\vec{k}}{(2\pi)^4} \tilde{B}_{\mu}(\vec{k},\omega)\, \hat{k} \times \vec{\epsilon}_{\mu}(\vec{k})\, e^{i(\vec{k}\cdot\vec{x} - \omega t)}\, \delta(\omega^2 - c^2|\vec{k}|^2)$$

Note: $\delta(\omega^2 - c^2|\vec{k}|^2) = \delta(\omega - c|\vec{k}|) + \delta(\omega + c|\vec{k}|)$

$$\Rightarrow \vec{E}(\vec{x},t) = \sum_{\mu} \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{\tilde{E}_{\mu}(\vec{k}, \omega = c|\vec{k}|)}{2\pi} \hat{e}_{\mu}(\hat{k})\, e^{i(\vec{k}\cdot\vec{x} - \omega_k t)}$$

$$+ \sum_{\mu} \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{\tilde{E}_{\mu}(\vec{k}, \omega = -c|\vec{k}|)}{2\pi} \hat{\epsilon}_{\mu}(\hat{k})\, e^{i(\vec{k}\cdot\vec{x} + \omega_k t)}$$

$$\omega_k = c|\vec{k}|$$

Aside: $$\sum_\mu \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{\tilde{E}_\mu(\vec{k}, \omega = -c|\vec{k}|)}{(2\pi)} \hat{\epsilon}_\mu(\hat{k}) e^{i(\vec{k}\cdot\vec{x} + \omega_k t)}$$

$\vec{k} \to -\vec{k}$ $$= \sum_\mu \int \frac{d^3k}{(2\pi)^3} \frac{\tilde{E}_\mu(-\vec{k}, \omega = -c|\vec{k}|)}{(2\pi)} \hat{\epsilon}_\mu(-\hat{k}) e^{-i(\vec{k}\cdot\vec{x} - \omega_k t)}$$

$$= \sum_\mu \int \frac{d^3k}{(2\pi)^3} \frac{\tilde{E}^*_\mu(\vec{k}, \omega = c|\vec{k}|)}{(2\pi)} \hat{\epsilon}^*_\mu(\hat{k}) \left(e^{i(\vec{k}\cdot\vec{x} - \omega_k t)}\right)^*$$

$$\therefore \quad \vec{E}(\vec{x},t) = \vec{E}^{(+)}(\vec{x},t) + \vec{E}^{(-)}(\vec{x},t)$$

"Positive freq Component" "Negative freq. component"

$$\vec{E}^{(+)}(\vec{x},t) \equiv \sum_\mu \int \frac{d^3k}{(2\pi)} \tilde{E}_\mu(\vec{k}) \hat{\epsilon}_\mu(\hat{k}) e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

$$\vec{E}^{(-)}(\vec{x},t) = \vec{E}^{(+)}(\vec{x},t)^*$$

Similarly for $\vec{B}$

One dimensional propagation for polarized field

$$\vec{E}(z,t) = \hat{e} \int \frac{dk\,d\omega}{(2\pi)^2}\, \tilde{E}(k,\omega)\, e^{i(kz-\omega t)}\, \delta(\omega^2 - c^2 k^2)$$

(i)
$$= \hat{e} \int \frac{d\omega}{2\pi} \left[\frac{\tilde{E}(k=\frac{\omega}{c}, \omega)}{2\pi}\, e^{-i\omega(t-\frac{z}{c})} \right.$$

$$\left. + \frac{\tilde{E}(k=-\frac{\omega}{c}, \omega)}{2\pi}\, e^{-i\omega(t+\frac{z}{c})} \right]$$

(ii)
$$= \hat{e} \int \frac{dk}{2\pi} \left[\frac{\tilde{E}(k, \omega=ck)}{2\pi}\, e^{ik(z-ct)} \right.$$

$$\left. \frac{\tilde{E}(k, \omega=-ck)}{2\pi}\, e^{ik(z+ct)} \right]$$

(i) Given $E(z=0,t)$ + $\dot{E}(z=0,t)$

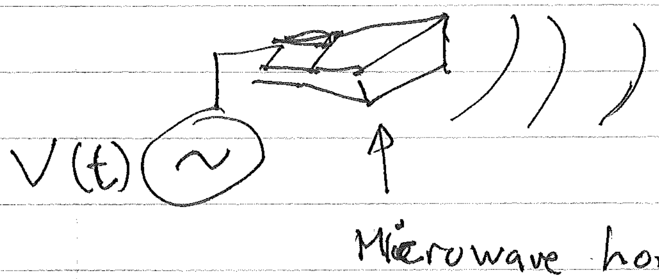
<u>Propagate in z</u> away from $z=0$

(ii) Given $E(z,t=0)$ + $\frac{\partial E}{\partial z}(z,t=0)$

<u>Progate in time</u> away from $t=0$

L#13.8

(i) Signal generator function of t



(ii) Given pulse as function of z; evolve in t

<u>Note:</u> Unlike a wave packet of a massive particle in free space propagating according to the Schrödinger equation, an electromagnetic pulse <u>does not spread</u>

<u>General expression for energy + momentum in field</u>

$$\vec{E}(\vec{x},t) = \sum_{\mu} \int \frac{d^3k}{(2\pi)^3} \left(E_\mu(\vec{k})\, \hat{e}_\mu(\hat{k})\, e^{i(\vec{k}\cdot\vec{x} - \omega_k t)} + \text{c.c.} \right)$$

$$\vec{B}(\vec{x},t) = \text{"} \quad \left(E_\mu(\vec{k})\, \hat{k} \times \hat{e}_\mu(\hat{k}) \quad + \text{c.c.} \right)$$

<u>Energy in field</u> $\langle U \rangle = \frac{1}{8\pi} \int d^3x \left\langle \left(|\vec{E}|^2 + |\vec{B}|^2 \right) \right\rangle$

$$\langle \vec{E}\cdot\vec{E} \rangle = 2\langle \vec{E}^{(+)}\cdot\vec{E}^{(-)} \rangle + \cancel{\langle \vec{E}^{(+)}\cdot\vec{E}^{(+)} \rangle}^{\,0} + \cancel{\langle \vec{E}^{(-)}\cdot\vec{E}^{(-)} \rangle}^{\,0}$$

$$\langle U_E \rangle = \frac{1}{4\pi} \int d^3x \langle \vec{E}^{(+)}\cdot\vec{E}^{(-)} \rangle \underset{\text{Parseval}}{=} \frac{1}{4\pi} \int \frac{d^3k}{(2\pi)^3} |\tilde{E}_\mu(\vec{k})|^2$$

$$\langle U_B \rangle = \frac{1}{4\pi}\int d^3x \, \langle \vec{B}^{(+)} \cdot \vec{B}^{(-)} \rangle = \frac{1}{4\pi}\int \frac{d^3k}{(2\pi)^3} |\tilde{E}_\mu(\vec{k})|^2$$

$\Rightarrow$ $\underline{\text{Energy density}}$ / mode $= \dfrac{|\tilde{E}_\mu(\vec{k})|^2}{2\pi} = u_{\vec{k},\mu}$

Note: For one photon in the mode, volume V

$$u_{\vec{k},\mu} = \frac{\hbar \omega_k}{V}$$

$\Rightarrow$ Electric field strength of photon

$$\tilde{E}_\mu(\vec{k}) = \sqrt{\frac{2\pi \hbar \omega_k}{V}}$$

<u>Momentum</u> $\quad \vec{P} = \int \dfrac{\vec{S}}{c^2}\, d^3x \quad , \quad \vec{S} = \dfrac{c}{4\pi}(\vec{E}\times\vec{B})$

Plane wave

$$\langle \vec{S} \rangle_{\vec{k},\mu} = u_{\vec{k},\mu}\, c\, \hat{k}$$

$$\Rightarrow \langle \vec{P} \rangle = \sum_\mu \int \frac{d^3k}{(2\pi)^3} \frac{u_{\vec{k},\mu}}{c} \hat{k}$$

Note: From relativity $E^2 = p^2c^2 + (mc^2)^2$

Photon $m=0 \Rightarrow E = pc \Rightarrow p = E/c$

Momentum density/photon $= \dfrac{\hbar\omega_k/c\, \hat{k}}{V} = \hbar\vec{k}/V$