Lecture 15: Wave Propagation in Linear Media

Consider linear, uniform, isotropic, nondispersive case with no (free) sources

$$\rho_{free} = \vec{J}_{free} = 0 \quad , \quad \vec{D} = \epsilon \vec{E}, \quad \vec{B} = \mu \vec{H}$$

Maxwell's Eqns:

$$\vec{\nabla} \cdot \vec{D} = 0 \quad \Rightarrow \quad \vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla} \times \vec{B} = \frac{\mu \epsilon}{c} \frac{\partial \vec{E}}{\partial t}$$

Wave equation: $\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{1}{c} \frac{\partial}{\partial t} \vec{\nabla} \times \vec{B}$

$$\Rightarrow \left(\nabla^2 - \frac{\mu\epsilon}{c^2} \frac{\partial^2}{\partial t^2} \right) \begin{pmatrix} \vec{E} \\ \vec{D} \end{pmatrix} = 0$$

$$\left(\nabla^2 - \frac{\mu\epsilon}{c^2} \frac{\partial^2}{\partial t^2} \right) \begin{pmatrix} \vec{H} \\ \vec{B} \end{pmatrix} = 0$$

Same as wave equation in free space but $\frac{1}{c^2} \to \frac{\mu\epsilon}{c^2}$

Plane wave solutions

$$\vec{E} = \mathrm{Re} \left(\underset{\sim}{\vec{E}_0}\, e^{i(\vec{k} \cdot \vec{x} - \omega t)} \right)$$

$$\vec{H} = \mathrm{Re} \left(\underset{\sim}{\vec{H}_0}\, e^{i(\vec{k} \cdot \vec{x} - \omega t)} \right)$$

Plug plane wave ansatz into wave equation

$$\vec{\nabla} \Rightarrow i\vec{k} \qquad \frac{\partial}{\partial t} \Rightarrow -i\omega$$

$$\Rightarrow -|\vec{k}|^2 + \frac{\mu\epsilon}{c^2}\omega^2 = 0 \Rightarrow \boxed{\omega = \frac{c}{\sqrt{\mu\epsilon}}|\vec{k}|}$$

Dispersion relation

Phase velocity: $v_p \equiv \frac{\omega}{|\vec{k}|} = \frac{c}{\sqrt{\mu\epsilon}}$

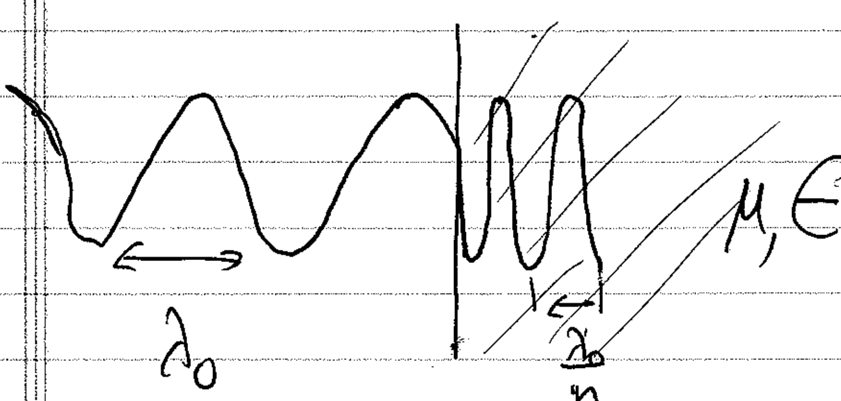
Index of refraction $\boxed{n \equiv \frac{c}{v_p} = \sqrt{\mu\epsilon}}$

Wave length inside medium

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{\omega}v_p = \frac{v_p}{\nu} = \underbrace{\left(\frac{c}{\nu}\right)}_{\lambda_0}\frac{1}{n}$$

$\boxed{\lambda = \frac{\lambda_0}{n}}$ Wave length shorter, ν same

Wave "bunches" up inside material due to slower phase velocity



Relation between $\underset{\sim}{\vec{E}_0}$, $\underset{\sim}{\vec{H}_0}$, $\vec{k}$

$$\vec{\nabla}\cdot\vec{E}=0, \quad \vec{\nabla}\cdot\vec{B}=0 \quad \Rightarrow \quad \boxed{\begin{array}{c}\vec{k}\cdot\underset{\sim}{\vec{E}_0}=0, \quad \vec{k}\cdot\underset{\sim}{\vec{H}_0}=0 \\ \text{transverse waves}\end{array}}$$

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} = -\frac{\mu}{c}\frac{\partial\vec{H}}{\partial t}$$

$$\Rightarrow i\vec{k}\times\underset{\sim}{\vec{E}_0} = i\mu\frac{\omega}{c}\underset{\sim}{\vec{H}_0}, \qquad |\vec{k}| = \frac{\omega}{v_p} = \frac{\omega}{c}\sqrt{\mu\epsilon}$$

$$\Rightarrow \underset{\sim}{\vec{H}_0} = \sqrt{\frac{\epsilon}{\mu}}\ \vec{e}_k\times\underset{\sim}{\vec{E}_0}$$

Define $\boxed{Z \equiv \frac{|\underset{\sim}{\vec{E}_0}|}{|\underset{\sim}{\vec{H}_0}|} = \sqrt{\frac{\mu}{\epsilon}} = \underline{\text{Wave impedence}}}$

like $\frac{\text{Voltage}}{\text{Current}}$ for AC circuit

<u>Energy flow:</u>

$$\langle u_{electric}\rangle = \frac{\epsilon}{8\pi}\langle\vec{E}^2\rangle = \frac{\epsilon}{8\pi}\left(\frac{\underset{\sim}{\vec{E}_0}^*\cdot\underset{\sim}{\vec{E}_0}}{2}\right) = \frac{\epsilon}{16\pi}|\underset{\sim}{\vec{E}_0}|^2$$

$$\langle u_{mag}\rangle = \frac{\mu}{8\pi}\langle\vec{H}^2\rangle = \frac{\mu}{8\pi}\left(\frac{\underset{\sim}{\vec{H}_0}^*\cdot\underset{\sim}{\vec{H}_0}}{2}\right)$$

$$= \frac{\mu}{16\pi}|\underset{\sim}{\vec{H}_0}|^2 = \frac{\mu}{16\pi}\frac{|\underset{\sim}{\vec{E}_0}|^2}{Z^2} = \frac{\epsilon}{16\pi}|\underset{\sim}{\vec{E}_0}|^2$$

$$= \langle u_{electric}\rangle$$

- Total energy density of plane wave in linear dielectric

$$\langle u \rangle = \langle u_{elec} \rangle + \langle u_{mag} \rangle = \frac{\epsilon}{8\pi} |\tilde{E}_0|^2$$

Intensity:

$$\langle \vec{S} \rangle = \frac{c}{4\pi} \langle \vec{E} \times \vec{H} \rangle = \frac{c}{4\pi} \mathrm{Re} \frac{(\tilde{E}_0^* \times \tilde{H}_0)}{2}$$

$$= \frac{c}{8\pi} \frac{|\vec{\tilde{E}}_0|^2}{Z} \vec{e}_k = \frac{c}{Z\epsilon} \left(\frac{\epsilon}{8\pi} |\vec{E}_0|^2 \right) \vec{e}_k$$

$$= \frac{c}{\sqrt{\mu\epsilon}} \langle u \rangle \vec{e}_k = v_p \langle u \rangle \vec{e}_k$$

$\Rightarrow$ In a plane wave, ~~[illegible]~~ non-dispersive linear dielectric, ~~[illegible]~~ energy flow at phase velocity

General case of Homogeneous medium - Conductor

Suppose medium has a finite conductivity.

Ohm's Law $\quad \vec{J}_{free} = \sigma \vec{E}$

Continuity eqn: $\frac{\partial}{\partial t} \rho_{free} = -\vec{\nabla} \cdot \vec{J}_{free} = -\sigma \vec{\nabla} \cdot \vec{E} = \frac{-\sigma}{4\pi\epsilon} \rho_{free}$

$\Rightarrow \rho_{free}(t) = e^{-\frac{\sigma}{4\pi\epsilon} t} \rho(0) \Rightarrow$ After time $t \sim \frac{4\pi\epsilon}{\sigma}$

$\rho_f \to 0$: Surface only

Maxwell's Eqns:

$$\vec{\nabla}\cdot\vec{E} = 0 , \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} , \qquad \vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\mu\sigma\vec{E} + \frac{\mu\epsilon}{c^2}\frac{\partial\vec{E}}{\partial t}$$

~~Modified W.E.~~

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\frac{1}{c}\vec{\nabla}\times\frac{\partial\vec{B}}{\partial t}$$

$$\Rightarrow -\nabla^2\vec{E} = -\frac{4\pi}{c^2}\mu\sigma\frac{\partial\vec{E}}{\partial t} - \frac{\mu\epsilon}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}$$

Modified wave equation

$$\boxed{\left(\nabla^2 - \frac{4\pi\mu\sigma}{c^2}\frac{\partial}{\partial t} - \frac{\mu\epsilon}{c^2}\frac{\partial^2}{\partial t^2}\right)\left\{\begin{matrix}\vec{E}\\ \vec{B}\end{matrix}\right\} = 0}$$

Plane wave solutions $\vec{E} = \mathrm{Re}\left(\underset{\sim}{E}_0\, e^{i(\underset{\sim}{k}\cdot\vec{x}-\omega t)}\right)$

$\vec{B} = \mathrm{Re}\left(\underset{\sim}{B}_0\, e^{i(\underset{\sim}{k}\cdot\vec{x}-\omega t)}\right)$ (Allow $\underset{\sim}{k}$ complex)

Dispersion relation: $-\underset{\sim}{k}^2 + i\frac{4\pi\mu\sigma\omega}{c^2} + \frac{\omega^2}{c^2}\mu\epsilon = 0$

$$\Rightarrow \boxed{\underset{\sim}{k} = \frac{\omega}{c}\sqrt{\mu\epsilon}\left(1 + i4\pi\frac{\sigma}{\epsilon\omega}\right)^{1/2}}$$ Complex wave number

Physical meaning: $\underset{\sim}{k} = k_R + i k_I$ (real and imaginary parts)

$$\Rightarrow \vec{E} = \mathrm{Re}\left(\vec{E}_0\, e^{i(\vec{k}_R\cdot\vec{x} - \omega t)}\, e^{-k_I(\hat{k}\cdot\vec{x})}\right)$$

Say $\underset{\sim}{E}_0$ real $\qquad \hat{k} = \hat{z}$

$$\Rightarrow \vec{E} = E_0\cos(k_R z - \omega t)\, e^{-k_I z}$$

(next page)

$\Rightarrow$ Real part of $\underset{\sim}{k}$ $\Rightarrow$ propagation

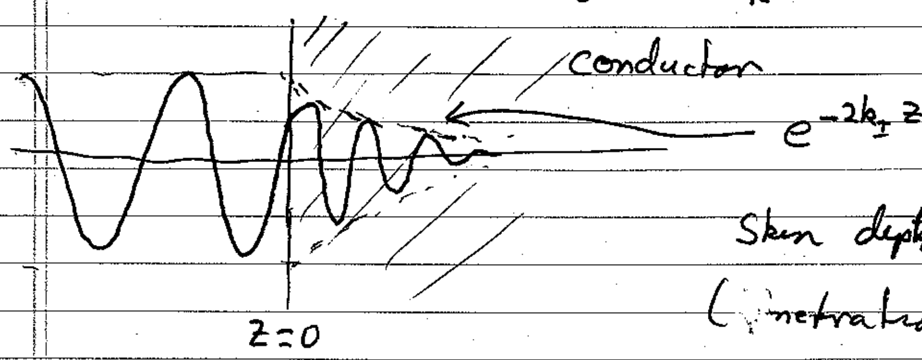
Phase velocity $v_p = \dfrac{\omega}{k_R}$

Imag part of $\underset{\sim}{k}$ $\Rightarrow$ absorption/attenuation

Intensity $\langle \vec{S} \rangle = \dfrac{1}{8\pi\mu} \mathrm{Re}\left(\underset{\sim}{\vec{E}} \times \underset{\sim}{\vec{B}}^*\right) \approx e^{-2k_I z} \langle \vec{S}(0) \rangle$

(see homework)

$$\vec{E}^2 = E_0^2 \cos^2(k_R z - \omega t)\, e^{-2k_I z}$$



Skin depth $\delta = \dfrac{1}{2k_I}$

(penetration depth)

Limits of $\underset{\sim}{k}$ for

- "Poor conductor" $\sigma \ll \epsilon \dfrac{\omega}{4\pi}$

$$\Rightarrow \underset{\sim}{k} = \frac{\omega}{c}\sqrt{\mu\epsilon}\left(1 + i\frac{4\pi\sigma}{\epsilon\omega}\right)^{1/2} \underset{\text{Taylor}}{\approx} \frac{\omega}{c}\sqrt{\mu\epsilon}\left(1 + i\frac{2\pi\sigma}{\epsilon\omega}\right)$$

$$\Rightarrow k_R = \frac{\omega}{c}\sqrt{\mu\epsilon}, \quad k_I = \frac{2\pi\sigma}{c}\sqrt{\frac{\mu}{\epsilon}}$$

- "Good conductor" $\sigma \gg \epsilon \dfrac{\omega}{4\pi}$

$$\underset{\sim}{k} = \frac{\omega}{c}\sqrt{\mu\epsilon}\left(1 + i\frac{4\pi\sigma}{\epsilon\omega}\right)^{1/2} \simeq \frac{\sqrt{4\pi\mu\sigma\omega}}{c}\,(i)^{1/2}$$

(Aside $(i)^{1/2} = \left(e^{i\pi/2}\right)^{1/2} = e^{i\pi/4} = \dfrac{1+i}{\sqrt{2}}$

$$\Rightarrow k_R = k_I = \frac{\sqrt{2\pi\mu\sigma\omega}}{c}$$