Lecture #16 Wave propagation in dispersive media

Simple models of dispersion

The constitutive relations we've assumed are naive

$$\vec{P}(\vec{x},t) = \chi_e \vec{E}(\vec{x},t), \quad \vec{M}(\vec{x},t) = \chi_m \vec{H}(\vec{x},t), \quad \vec{J}(\vec{x},t) = \sigma \vec{E}(\vec{x},t)$$

Assumes that the response of the medium to the field is instaneous

More realistic: χ_e, σ, χ_m represent steady state response coefficients to a drive at freq ω

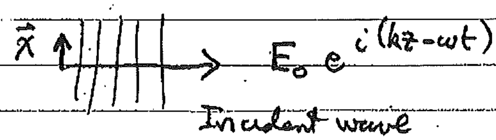
$\Rightarrow$ Local relation in the frequency domain

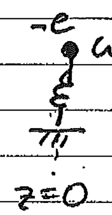
$$\underset{\sim}{\vec{P}}(\vec{x},\omega) = \tilde{\chi}_e(\omega)\, \underset{\sim}{\vec{E}}(\vec{x},\omega) \quad \text{in Fourier space}$$

$$\Rightarrow \vec{P}(\vec{x},t) = \int \frac{d\omega}{2\pi}\, \underset{\sim}{\vec{P}}(\vec{x},\omega)\, e^{-i\omega t} = \int dt'\, \underbrace{\chi_e(t-t')}_{\text{response function}}\, \vec{E}(\vec{x},t')$$

Lorentz Model of dielectrics

Bound charges in molecules modeled by charges on spring





(Charge on spring, ω_0 = natural resonance, Γ = damping coef.)

Eqn of motion:

$$m\ddot{x} = -m\omega_0^2 x - m\Gamma \dot{x} - eE(z=0,t)$$

$\Rightarrow$ Force SHO $\boxed{\ddot{x} + \Gamma\dot{x} + \omega_0^2 x = -\frac{e}{m}E(z=0,t)}$

Near resonance $\omega \approx \omega_0$, Define "detuning" $\Delta = \omega - \omega_0$

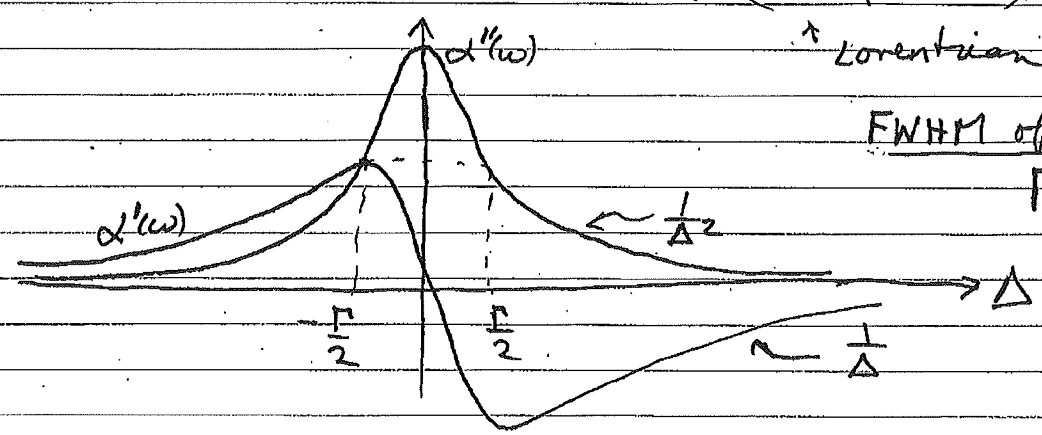
$$\omega_0^2 - \omega^2 = (\omega_0 + \omega)(\omega_0 - \omega) = (2\omega_0 + \Delta)(-\Delta)$$

$$\approx -2\omega_0 \Delta \qquad \Delta << \omega_0$$

$$\Rightarrow \tilde{\alpha}(\omega) \approx \left(\frac{e^2}{2m\omega_0}\right)\left(-\frac{1}{\Delta + i\frac{\Gamma}{2}}\right) \quad \text{Complex Lorentzian}$$

$$= \frac{e^2}{2m\omega_0}\left(\frac{-\Delta}{\Delta^2 + \frac{\Gamma^2}{4}} + i\frac{\Gamma/2}{\left(\Delta^2 + \frac{\Gamma^2}{4}\right)}\right)$$

↑ Lorentzian



FWHM of Lorentzian Γ

Given N $\frac{\text{molecules}}{\text{Volume}}$, Z ~~[illegible]~~ electrons/molecule,

multiple resonances ω_j, Γ_j , f_j $\frac{\text{electrons}}{\text{molecule}}$ contribute

Total polarization: $\tilde{P}(\omega) = \sum_j N f_j \, \tilde{p}_j(\omega)$ (ignoring local-field corrections)

$$= \left[\frac{Ne^2}{m} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\Gamma_j}\right] \tilde{E}(\omega)$$

Complex susceptibility $\tilde{\chi}_e(\omega) = \sum_j N \tilde{\alpha}_j(\omega)$

Physically: Why $\mathrm{Im}(\tilde{\chi}) \Rightarrow$ Absorption

Steady state polarization $\bar{P}(t) = \mathrm{Re}(\tilde{P}\, e^{-i\omega t})$

$$\underline{E} = \mathrm{Re}(\underset{\uparrow}{E_0}\, e^{-i\omega t}) = E_0 \cos\omega t \quad \Rightarrow \quad P(t) = \mathrm{Re}(\tilde{\chi}_e E_0\, e^{-i\omega t})$$

take real

$$\Rightarrow P(t) = \mathrm{Re}(\tilde{\chi}) E_0 \cos\omega t + \mathrm{Im}(\tilde{\chi}) E_0 \sin\omega t$$

Polarization current: $J = \frac{\partial P}{\partial t} = \omega\left(-\mathrm{Re}(\tilde{\chi}) E_0 \sin\omega t + \mathrm{Im}(\tilde{\chi}) E_0 \cos\omega t\right)$

Ohmic dissipation: Rate of work done by fields on charges / volume $\vec{J}\cdot\vec{E}$

$$\Rightarrow \langle J(t) E(t)\rangle = \langle J(t) E_0 \cos\omega t\rangle = \frac{1}{2}\omega\, \mathrm{Im}(\tilde{\chi}) E_0^2$$

$\Rightarrow$ In quadrature component of $P(t) \Rightarrow$ In phase component of J

Using complex notation:

$$\langle JE\rangle = \frac{1}{2}\mathrm{Re}(\tilde{J}^* \tilde{E}) = \frac{1}{2}\mathrm{Re}(-i\omega\, \tilde{P}^* \tilde{E})$$

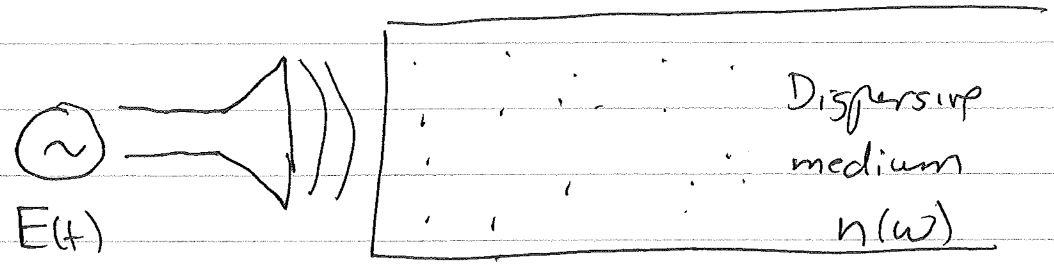
$$= \frac{\omega}{2}|E|^2\, \mathrm{Re}(-i\tilde{\chi}) = \frac{\omega}{2}\mathrm{Im}(\tilde{\chi}) E_0^2$$

$\mathrm{Im}(\tilde{\chi}) > 0 \Longrightarrow$ Absorption (Passive)

$\mathrm{Im}(\tilde{\chi}) < 0 \Longrightarrow$ Gain! (Laser) (Active)

Propagation and dispersion

Suppose we inject a signal E(t) at the input face of some medium



Input signal $E(t) = \int_{-\infty}^{\infty} \tilde{E}(\omega)\, e^{-i\omega t} \frac{d\omega}{2\pi}$

$$\Rightarrow E(t) = \underbrace{\int_0^{\infty} \frac{d\omega}{2\pi} \tilde{E}(\omega)\, e^{-i\omega t}}_{E^{(+)}(t)} + \underbrace{\int_{-\infty}^{0} \frac{d\omega}{2\pi} \tilde{E}(\omega)\, e^{-i\omega t}}_{E^{(-)}(t)}$$

$E^{(+)}(t)$: positive frequency component

$E^{(-)}(t)$: negative frequency component

Recall $\left(E^{(+)}(t)\right)^* = \int_0^{\infty} \frac{d\omega}{2\pi} \tilde{E}^*(\omega)\, e^{+i\omega t} = \int_0^{\infty} \frac{d\omega}{2\pi} \tilde{E}(-\omega) \frac{e^{+i\omega t}}{2\pi}$

$$= \int_{-\infty}^{0} \frac{d\omega}{2\pi} \tilde{E}(\omega) \frac{e^{-i\omega t}}{2\pi} = E^{(-)}(t)$$

$$\Rightarrow E(t) = 2\,\mathrm{Re}\left(E^{(+)}(t)\right)$$

Propagation: Boundary condition $E(z=0,t) = E(t)$

$$\Rightarrow E(z=0,t) = 2\,\mathrm{Re}\left[\int_0^\infty \tilde{E}(\omega)\, e^{-i\omega t}\, \frac{d\omega}{2\pi}\right]$$

Each frequency propagates plane wave with wave fronts moving at phase velocity

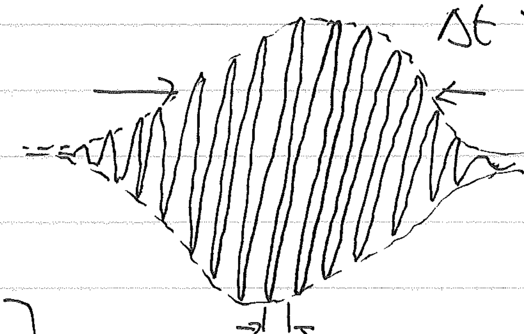
$$\Rightarrow E(z,t) = 2\,\mathrm{Re}\left[\int_0^\infty \tilde{E}(\omega)\, e^{-i\omega\left(t - \frac{z}{v_p(\omega)}\right)}\, \frac{d\omega}{2\pi}\right]$$

$$v_p(\omega) = \frac{\omega}{k(\omega)} = \frac{c}{n(\omega)}$$

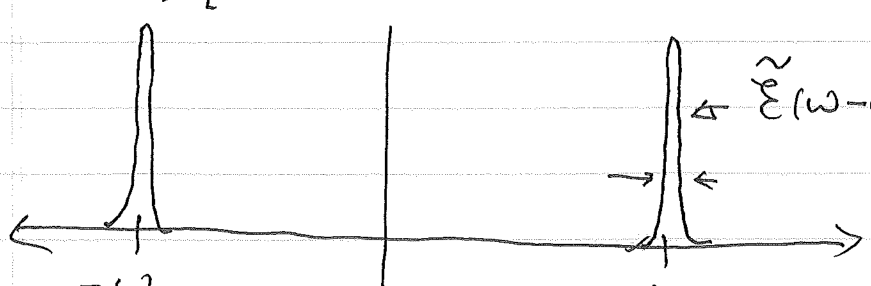
$$\Rightarrow E(z,t) = 2\,\mathrm{Re}\Big[\underbrace{\int_0^\infty \tilde{E}(\omega)\, e^{i(k(\omega)z - \omega t)}}_{E^{(+)}(z,t)}\, \frac{d\omega}{2\pi}\Big]$$

Suppose the input signal is a quasimonochromatic pulse

$$E(t) = \underset{\substack{\uparrow \\ \text{slowly varying} \\ \text{envelope}}}{\mathcal{E}(t)}\ \underset{\substack{\uparrow \\ \text{carrier} \\ \text{wave}}}{\cos\omega_0 t}$$



$$\Rightarrow \tilde{E}(\omega) = \frac{1}{2}\left[\tilde{\mathcal{E}}(\omega-\omega_0) + \tilde{\mathcal{E}}(\omega+\omega_0)\right]$$



In the quasimonochromatic approximation with $\Delta\omega \ll \omega_0$

$$E^{(+)}(t) \approx \frac{1}{2}\int_0^\infty \tilde{\mathcal{E}}(\omega-\omega_0)\, e^{-i\omega t}\, \frac{d\omega}{2\pi}$$

$$\approx \frac{1}{2}\int_{-\infty}^\infty \tilde{\mathcal{E}}(\omega-\omega_0)\, e^{-i\omega t}\, \frac{d\omega}{2\pi}$$

$$\Rightarrow E^{(+)}(z,t) \approx \frac{1}{2}\int_{-\infty}^\infty \frac{d\omega}{2\pi}\, \tilde{\mathcal{E}}(\underbrace{\omega-\omega_0}_{\equiv\delta})\, e^{i(k(\omega)z - \omega t)}$$

$$= \frac{1}{2}\left[\int_{-\infty}^\infty \frac{d\delta}{2\pi}\, \tilde{\mathcal{E}}(\delta)\, e^{i(k(\omega_0+\delta)z - \delta)t}\right] e^{-i\omega_0 t}$$

<u>Approximation</u>

$\tilde{\mathcal{E}}(\delta)$ has support at $\delta = 0$ around a small band width $\Delta\omega \Rightarrow$ Taylor expand $k(\omega_0+\delta)$ around $\delta = 0$

$$\Rightarrow k(\omega_0+\delta) \approx k(\omega_0) + \delta \left.\frac{dk}{d\omega}\right|_{\omega_0} + \frac{1}{2}\delta^2 \left.\frac{d^2k}{d\omega^2}\right|_{\omega_0} + \dots$$

where $k(\omega_0) = \frac{\omega_0}{c} n(\omega_0) \equiv k_0$

To first order in δ

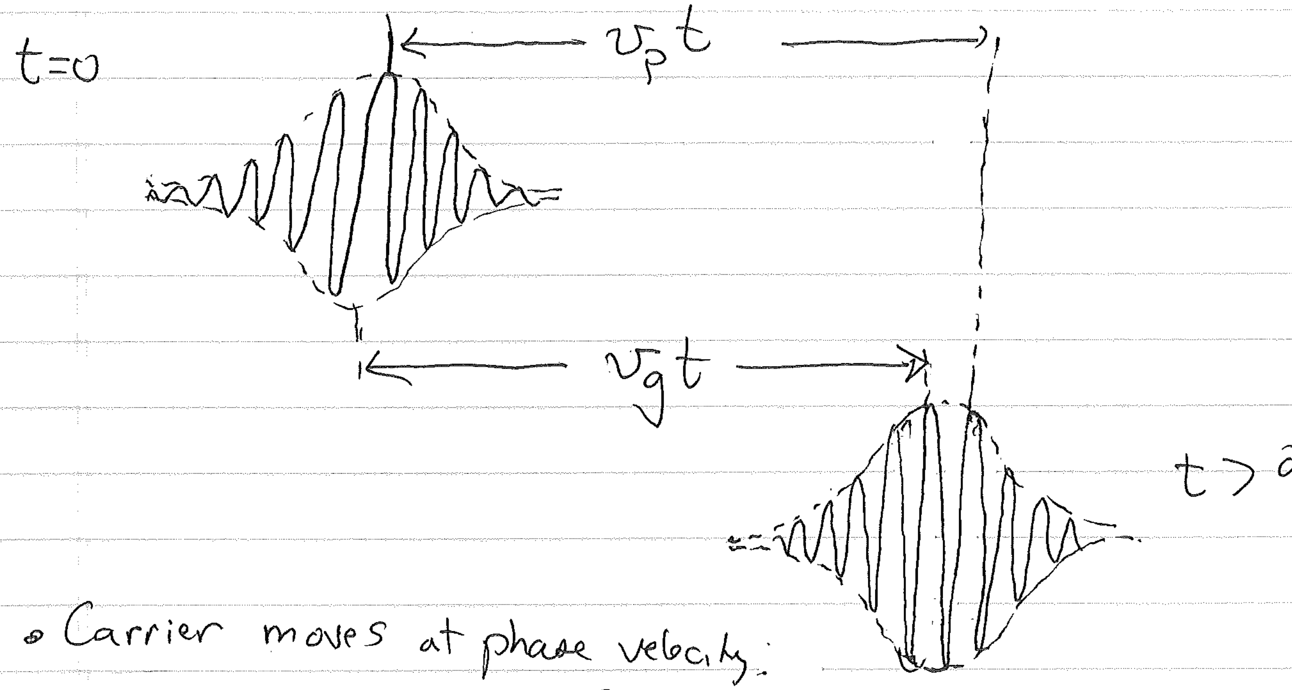
$$E^{(+)}(z,t) \cong \left[\frac{1}{2}\int_{-\infty}^\infty \frac{d\delta}{2\pi}\, \tilde{\mathcal{E}}(\delta)\, e^{-i\delta\left(t - \left.\frac{dk}{d\omega}\right|_{\omega_0} z\right)}\right] e^{i(k_0 z - \omega_0 t)}$$

Define the group velocity: $v_g = \left.\frac{d\omega}{dk}\right|_{\omega_0}$

$$\Rightarrow E^{(+)}(z,t) = \left[\frac{1}{2}\underbrace{\int_{-\infty}^{\infty}\frac{d\delta}{2\pi}\,\tilde{\mathcal{E}}(\delta)\, e^{-i\delta\left(t-\frac{z}{v_g}\right)}}_{\mathcal{E}\left(t-\frac{z}{v_g}\right)}\right] e^{i(k_0 z-\omega_0 t)}$$

$\Rightarrow$ To first order in δ

$$\boxed{E(z,t) = 2\,\mathrm{Re}\left[E^{(+)}(z,t)\right] = \mathcal{E}\left(t-\frac{z}{v_g}\right)\cos(k_0 z-\omega_0 t)}$$



- Carrier moves at phase velocity:

$$v_p = \frac{\omega_0}{k_0} = \frac{c}{n(\omega_0)}$$

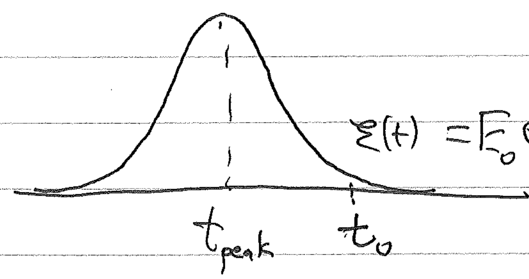
- Peak of pulse envelope moves at the group velocity:

$$v_g = \left.\frac{d\omega}{dk}\right|_{\omega_0}$$

In a region with $\mathrm{Im}(\tilde{n}(\omega)) \approx 0$, normal dispersion $v_g < c$. The phase velocity $v_p = \frac{c}{n(\omega)}$ can be $> c$ when $n(\omega) < 1$.

But, the "carrier wave" carries no information — infinite wave train with no begining or end — so there is no violation of Einstein-causality.

Does information travel at the group velocity? No, for a perfectly analytic function $\mathcal{E}(t)$, eg. Gaussian, all derivatives are smooth $\Rightarrow$ Taylor Series expansion predicts the entire function!



$$\mathcal{E}(t) = E_0 e^{-\frac{(t-t_{peak})^2}{2\sigma^2}}$$

$$\mathcal{E}(t) = \sum \frac{(t-t_0)^n}{n!} \underbrace{\left.\frac{d^n\mathcal{E}}{dt^2}\right|_{t_0}}$$

derivatives @ t_0 encode the time of arrival of peak

$\Rightarrow$ No information encoded in perfectly smooth function.

Information $\Rightarrow$ ~~Meaning~~ learning something new we didn't already know

Effect of higher order terms in Taylor series expansion of $k(\omega)$: Dispersion

$$\frac{d^2k}{d\omega^2} = \frac{d}{d\omega}\frac{1}{v_g} = \frac{dv_g}{d\omega}\frac{1}{-v_g^2} \qquad \text{G.V.D.}$$

(Group velocity dispersion)

$$E(z,t) \cong \mathrm{Re}\left[\int_{-\infty}^{\infty}\left(\frac{d\Omega}{2\pi}\tilde{\mathcal{E}}(\Omega)\, e^{-i\Omega(t-\frac{z}{v_g})}\, e^{\frac{-ik''\Omega^2}{2}}\right)e^{i(k_0z-\omega_0t)}\right]$$

quadratic dependence

For special case of a Gaussian input pulse output is also Gaussian with pulse duration

$$\Delta t(z) = \sqrt{\Delta t^2(0) + \left(\frac{k_0''}{\Delta t(0)}\right)^2 z^2}$$

Dispersion length : $\frac{\Delta t(0)}{k_0''}$

General arguement : Due to finite bandwidth $\Delta\omega$

there will be a spread of group velocities $\Delta v_g = \frac{dv_g}{d\omega}\Delta\omega$

$\Rightarrow$ Spread of arrival times :

$$\Delta t(z) = \Delta\left(\frac{z}{v_g}\right) = z\left(\frac{-\Delta v_g}{v_g^2}\right) = -z\,\frac{\frac{dv_g}{d\omega}}{v_g^2}\Delta\omega$$

$$= -k''z\,(\Delta\omega) \qquad \text{But } \Delta\omega \sim \frac{1}{\Delta t_0}$$

Add to initial spread in squar

$$\Delta t(z)^2 = \Delta t_0^2 + \frac{k''^2}{\Delta t_0^2} z^2$$