Lecture 18 Radiation by prescribed sources

Inhomogenous wave equation (scalar field)

$$\underbrace{\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)}_{\Box \,\equiv\, \text{d'Alembertian}} \psi(\vec{x},t) = \underset{\substack{\Uparrow \\ \text{Source function}}}{s(\vec{x},t)}$$

General solution: $\psi(\vec{x},t) = \psi_{\text{Homogenous}}(\vec{x},t) + \psi_{\text{Particular}}(\vec{x},t)$

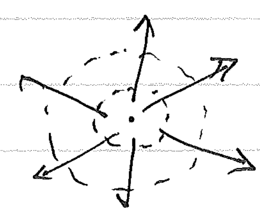
Linear response theory:

$$\psi_{\text{part}}(\vec{x},t) = \int d^3x'\, dt'\; G(\vec{x}-\vec{x}';\, t-t')\; s(\vec{x}';t')$$

$G(\vec{X},\tau)$: Greens function of wave eqn.

$\Box G = \delta^{(3)}(\vec{x})\,\delta(t)$: Wave generated by an impulse at the origin

Drop a pebble in a pond:



Spherical wave in 3D!

General source $s(\vec{x},t) = \int d^3x'\, dt'\; \underbrace{\delta^{(3)}(\vec{x}-\vec{x}')\,\delta(t-t')}_{\Downarrow}\; s(\vec{x}',t')$

Superposition of impulses $\Downarrow$ Total wave

$$\Downarrow$$

$$\psi(\vec{x},t) = \int d^3x'\, dt'\; G(\vec{x}-\vec{x}';\, t-t')\; s(\vec{x}',t')$$

Solving for the Green's function

But taking the Fourier Transform, a differential equation becomes an algebraic equation. We can then solve for the Greens function in Fourier space and inverse-Fourier to get it in space-time.

$$\tilde{G}(\vec{k},\omega) = \int d^3X\, d\tau \; G(\vec{X},\tau)\, e^{-i(\vec{k}\cdot\vec{X}-\omega\tau)}$$

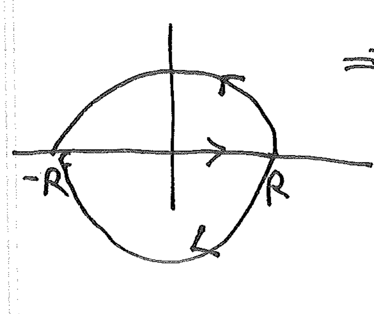
$$\Box\, G(\vec{X},\tau) = \delta^{(4)}(\vec{X},\tau) \iff \left(-k^2+\frac{\omega^2}{c^2}\right)\tilde{G}(\vec{k},\omega) = 1$$

$$\Rightarrow \quad \tilde{G}(\vec{k},\omega) = \frac{1}{\frac{\omega^2}{c^2}-k^2} = \frac{c^2}{(\omega+ck)(\omega-ck)}$$

<u>Invert:</u> $$G(\vec{X},\tau) = \int \frac{d^3k\, d\omega}{(2\pi)^4}\; \tilde{G}(\vec{k},\omega)\, e^{i(\vec{k}\cdot\vec{X}-\omega\tau)}$$

To perform the integral we will use the tricks of contour integration on the complex plane

<u>Aside:</u> Let $$f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, f(\omega)\, e^{-i\omega t}$$



$$\Rightarrow f(t) = \begin{cases} \lim \oint \frac{d\tilde{\omega}}{2\pi} f(\tilde{\omega})\, e^{-i\tilde{\omega}t} & t<0 \\ \oint \frac{d\tilde{\omega}}{2\pi} f(\tilde{\omega})\, e^{-i\tilde{\omega}t} & t>0 \end{cases}$$

$$\Rightarrow f(t) = \begin{cases} 2\pi i \sum \text{Residues in upper } \tilde{\omega} \text{ plane} \\ -2\pi i \sum \text{ " " lower " "} \end{cases}$$

For our problem

$$G(\vec{x},\tau) = \frac{c^2}{(2\pi)^4} \int d^3k \, e^{i\vec{k}\cdot\vec{x}} \int_{-\infty}^{+\infty} d\omega \, \frac{e^{-i\omega\tau}}{(\omega+ck)(\omega-ck)}$$

There are two poles : $\omega = \pm ck$ on <u>real axis</u>.
Thus, the causality of $G(\vec{x},\tau)$ not set but <u>imposed</u> by physical consideration.

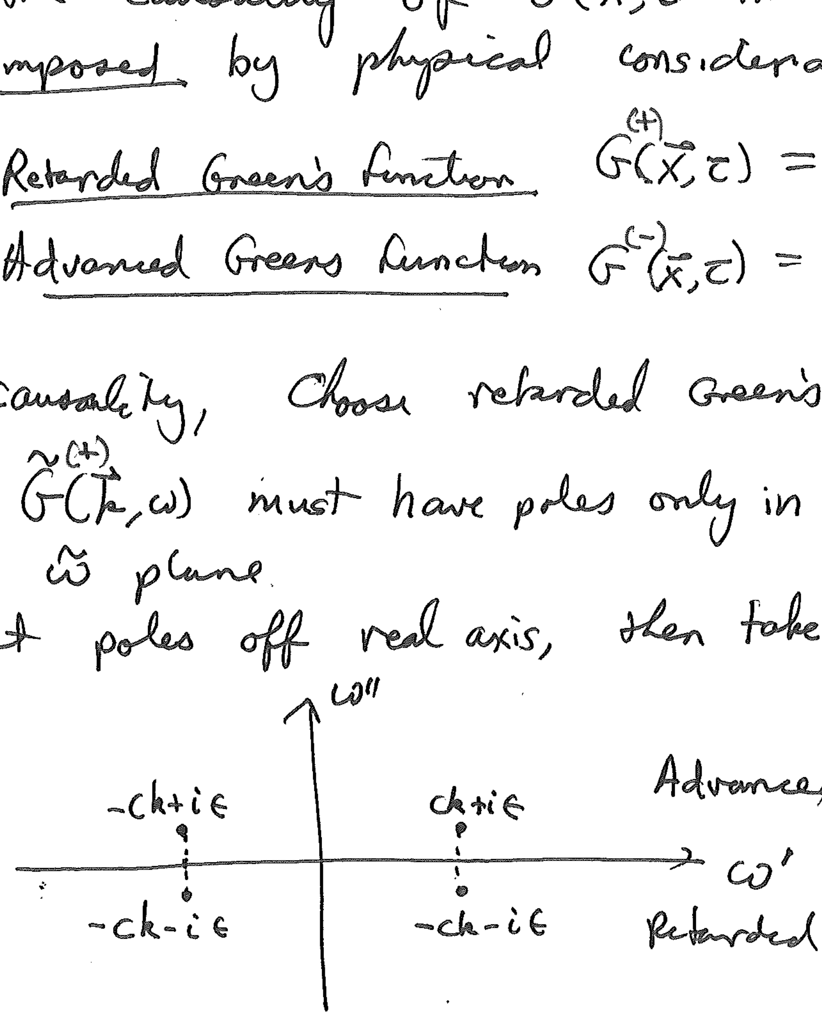
Define: <u>Retarded Green's function</u> $G^{(+)}(\vec{x},\tau) = 0 \quad \tau < 0$

<u>Advanced Greens function</u> $G^{(-)}(\vec{x},\tau) = 0 \quad \tau > 0$

For causality, Choose retarded Green's function

$\Rightarrow \tilde{G}^{(+)}(\vec{k},\omega)$ must have poles only in lower $\tilde{\omega}$ plane.

$\Rightarrow$ Must poles off real axis, then take limit



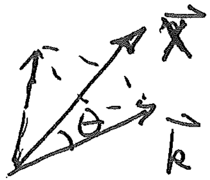
limit as $\epsilon \to 0$

$$\Rightarrow G^{(+)}(\vec{X},\tau) = \lim_{\epsilon\to 0} \frac{c^2}{(2\pi)^4}\int d^3k\, e^{i\vec{k}\cdot\vec{X}} \int_{-\infty}^{\infty} \frac{e^{-i\omega\tau}}{(\omega+ck+i\epsilon)(\omega-ck+i\epsilon)}$$

$$\Rightarrow G^{(+)}(\vec{X},\tau) = \begin{cases} 0 & t<0 \\ \frac{c^2}{(2\pi)^4}\int d^3k\, e^{i\vec{k}\cdot\vec{X}} \left\{(-2\pi i)\left(\frac{e^{-ick\tau}}{2ck} + \frac{e^{ick\tau}}{-2ck}\right)\right\} & t>0 \end{cases}$$

$\therefore$ for $t>0$
$$G_R(\vec{X},\tau) = -\frac{c}{(2\pi)^3}\underbrace{\int d^3k\, \frac{\sin(ck\tau)}{k}\, e^{ik\vec{X}}}_{\equiv \mathcal{I}}$$

<u>Use Spherical Symmetry</u>



$\vec{X}$ fixed "polar" axis

$$d^3k = k^2\, dk\, d\Omega_k$$

$$d\Omega_k = \sin\theta\, d\theta\, d\phi = -d\mu\, d\phi$$
$$\mu = \cos\theta$$

$$\Rightarrow \mathcal{I} = \int d^3k\, \frac{\sin(ck\tau)}{k}\, e^{i\vec{k}\cdot\vec{X}} = \int_0^\infty dk\; k\sin(ck\tau) \int d\Omega_k\, e^{ikr\cos\theta}$$

<u>Aside</u>: $\int d\Omega_k\, e^{ikr\cos\theta} = 2\pi\int_{-1}^{1} d\mu\, e^{i\mu kr} = \frac{2\pi}{ikr}\left(e^{ikr} - e^{-ikr}\right)$

$$\Rightarrow \mathcal{I} = \frac{-\pi}{r}\int_0^\infty dk \left(e^{ick\tau} - e^{-ick\tau}\right)\left(e^{ikr} - e^{-ikr}\right)$$

$$= +\frac{\pi}{r}\int_0^\infty dk \Big[e^{ik(r-c\tau)} + e^{-ik(r-c\tau)}$$

$$- e^{ik(r+c\tau)} - e^{-ik(r+c\tau)} \Big]$$

$$\Rightarrow d = \frac{\pi}{r} \int_{-\infty}^{\infty} dk \left[e^{ik(r-c\tau)} - e^{ik(r+c\tau)} \right]$$

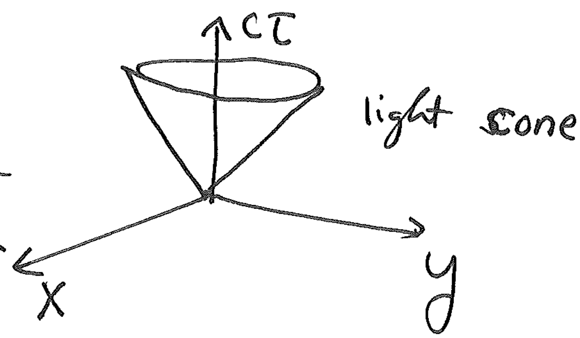
$$= \frac{\pi}{r} \left(2\pi\, \delta(r-c\tau) - 2\pi\, \cancel{\delta(r+c\tau)} \right)$$

$\rightarrow 0$ since $r>0$, $\tau>0$

$$\therefore \boxed{G^{(+)}(\vec{x},\tau) = -\frac{c}{4\pi|\vec{x}|}\,\delta(|\vec{x}|-c\tau) = -\frac{1}{4\pi}\,\frac{\delta\left(\tau - \frac{|\vec{x}|}{c}\right)}{|\vec{x}|}}$$

Space time diagram

Spherical wave front emiinating from origin

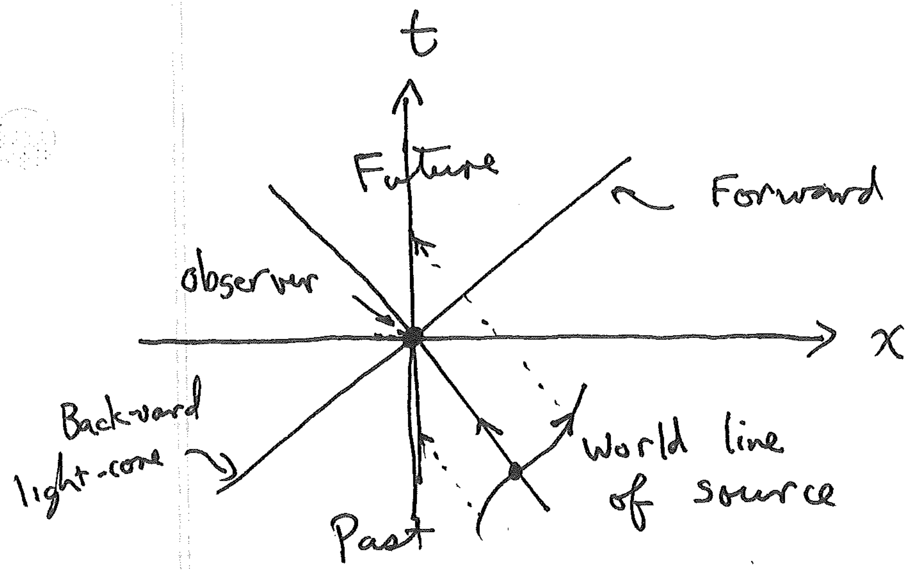


$$\Rightarrow \psi(\vec{x},t) = \int d^3x'\, dt'\, G^{(+)}(\vec{x}-\vec{x}', t-t')\, \mathcal{S}(\vec{x}', t')$$

$$= \frac{-1}{4\pi} \int d^3x'\, dt'\, \delta\left(t-t'-\frac{|\vec{x}-\vec{x}'|}{c}\right) \frac{\mathcal{S}(\vec{x}',t)}{|\vec{x}-\vec{x}'|}$$

$$\Rightarrow \psi(\vec{x},t) = -\frac{1}{4\pi} \int d^3x'\, \frac{\mathcal{S}(\vec{x}', t_{ret}(\vec{x}-\vec{x}', t))}{|\vec{x}-\vec{x}'|}$$

Retarded time: $t_{ret} = t - \frac{|\vec{x}-\vec{x}'|}{c}$ = local time − progaton time from $\vec{x} \rightarrow \vec{x}'$



Influences at any local position must have crossed my backward light cone

<u>Example</u>: Sinusoidally oscillating source $S(\vec{x},t) = \tilde{S}(\vec{x})\, e^{-i\omega t}$

$$\Rightarrow \Psi(\vec{x},t) = -\frac{1}{4\pi}\int S(\vec{x}')\, \frac{e^{-i\omega\left(t - \frac{|\vec{x}-\vec{x}'|}{c}\right)}}{|\vec{x}-\vec{x}'|}\, d^3x'$$

$$= \underbrace{\left(-\frac{1}{4\pi}\int S(\vec{x}')\, \frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}\right)}_{\tilde{\Psi}(\vec{x})}\, e^{-i\omega t}$$

$\tilde{\Psi}(\vec{x})$ = Superposition of spherical wave eminating from $\vec{x}'$ weighted by strength of source @ $\vec{x}'$

Potential Formulation & Gauge Transformations

Maxwell's Eqns:

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \qquad \vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \vec{B} = \vec{\nabla}\times\vec{A}$$

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \Rightarrow \vec{\nabla}\times\left(\vec{E} + \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right) = 0$$

$$\Rightarrow \vec{E} + \frac{1}{c}\frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}\phi$$

$$\Rightarrow \boxed{\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}, \quad \vec{B} = \vec{\nabla}\times\vec{A}}$$

Gauge invariance: $\boxed{\vec{A} \to \vec{A} + \vec{\nabla}\Lambda}$ $\quad \phi \Rightarrow \phi'$

$$\vec{E} \Rightarrow -\vec{\nabla}\phi' - \frac{1}{c}\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\left(\frac{1}{c}\frac{\partial\Lambda}{\partial t}\right)$$

$$\Rightarrow \phi' = \phi - \frac{1}{c}\frac{\partial\Lambda}{\partial t}$$

$$\boxed{\phi \to \phi - \frac{1}{c}\frac{\partial\Lambda}{\partial t}}$$

Lorentz Gauge: $\boxed{\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0}$

Coulomb Gauge $\boxed{\vec{\nabla}\cdot\vec{A} = 0}$

Always ~~possible~~

Dynamical equations

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial}{\partial t}\left(-\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\right)$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \frac{4\pi}{c}\vec{J} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}\left(\frac{\partial\phi}{c\,\partial t}\right)$$

$$\Rightarrow \boxed{\Box\vec{A} = -\frac{4\pi}{c}\vec{J} + \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}\right)}$$

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \Rightarrow \vec{\nabla}\cdot\left(-\vec{\nabla}\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right) = 4\pi\rho$$

$$\Rightarrow \boxed{\nabla^2\phi = -4\pi\rho + \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A})}$$

Lorentz Gauge

$$\boxed{\begin{aligned}\Box\vec{A} &= -\frac{4\pi}{c}\vec{J}\\ \Box\phi &= -4\pi\rho\end{aligned}} \Rightarrow \boxed{\begin{aligned}\vec{A}(\vec{x},t) &= \frac{1}{c}\int d^3x' \,\frac{\vec{J}(\vec{x}', t_{ret})}{|\vec{x}-\vec{x}'|}\\ \phi(\vec{x},t) &= \int d^3x' \,\frac{\rho(\vec{x}', t_{ret})}{|\vec{x}-\vec{x}'|}\end{aligned}}$$

Coulomb Gauge

$$\begin{aligned}\Box\vec{A} &= -\frac{4\pi}{c}\vec{J}_T\\ \nabla^2\phi &= -4\pi\rho\end{aligned} \Rightarrow \left\{\begin{aligned}\vec{A}(\vec{x},t) &= \frac{1}{c}\int d^3x' \,\frac{\vec{J}_T(\vec{x}', t_{ret})}{|\vec{x}-\vec{x}'|}\\ \phi(\vec{x},t) &= \int d^3x' \,\frac{\rho(\vec{x}', t)}{|\vec{x}-\vec{x}'|}\end{aligned}\right.$$

$\vec{J}_T$ is the transverse part of $\vec{J}$ (i.e. part with zero divergence)

$$\vec{J}_T = \vec{J} + \frac{1}{4\pi}\frac{\partial}{\partial t}(\vec{\nabla}\phi) \Rightarrow \vec{\nabla}\cdot\vec{J}_T = \vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0$$