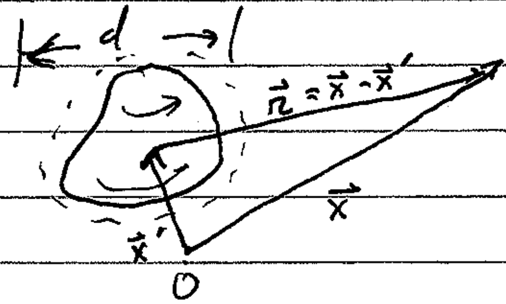
Lecture #18: Multipole Radiation: (Electric Dipole Radiation)

In the Lorentz Gauge: $\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0$

$$\Rightarrow \Box\vec{A} = -\frac{4\pi}{c}\vec{J}, \quad \Box\phi = -4\pi\rho$$

$$\Rightarrow \boxed{\vec{A}(\vec{x},t) = \frac{1}{c}\int d^3x' \frac{\vec{J}(\vec{x}', t_{ret})}{|\vec{x}-\vec{x}'|}, \quad \phi(\vec{x},t) = \int d^3x' \frac{\rho(\vec{x}', t_{ret})}{|\vec{x}-\vec{x}'|} \qquad t_{ret} = t - \frac{|\vec{x}-\vec{x}'|}{c}}$$

Given localized, slowly varying source
(we will quantify this)



Charteristic time scale for change in sources: T
(e.g. $\omega \sim \frac{1}{T}$)

Multipole expansion as in statics: $d \ll r$

$$|\vec{x}-\vec{x}'| \approx r - \hat{r}\cdot\vec{x}', \quad \frac{1}{|\vec{x}-\vec{x}'|} \approx \frac{1}{r} + \frac{\hat{r}\cdot\vec{x}'}{r^2}$$

Slowly varying source $\quad \frac{d}{c} \ll T \quad$ (Time to propagate across source $\ll$ time scale of change of source)

If $T \sim \frac{1}{\omega} \Rightarrow d \ll \frac{c}{\omega} = \frac{\lambda}{2\pi} \qquad \boxed{d \ll \lambda}$

$$\Rightarrow t_{ret} \approx t - \frac{|\vec{x}-\vec{x}'|}{c} = \underbrace{t - \frac{r}{c}}_{\text{Retarded time from O: } t_0} + \underbrace{\frac{\hat{r}\cdot\vec{x}'}{c}}_{\text{Extra time of propagation}}$$

Given slowly varying function localized to small $\vec{x}'$

$$f(t_{ret}) = f\left(t_0 + \frac{\hat{r}\cdot\vec{x}'}{c} + \cdots\right)$$

Taylor series $\approx f(t_0) + \frac{\hat{r}\cdot\vec{x}'}{c}\dot{f}(t_0) + \cdots$

$$\Rightarrow \phi(\vec{x},t) = \int d^3x' \left(\frac{1}{r} + \frac{\hat{r}\cdot\vec{x}'}{r^2} + \cdots\right)\left(\rho(\vec{x}',t_0) + \frac{\hat{r}}{c}\cdot\vec{x}'\,\dot{\rho}(\vec{x}',t_0) + \cdots\right)$$

$$\vec{A}(\vec{x},t) = \int d^3x' \left(\frac{1}{r} + \frac{\hat{r}\cdot\vec{x}'}{r^2} + \cdots\right)\left(\vec{J}(\vec{x}',t_0) + \frac{\hat{r}\cdot\vec{x}'}{c}\dot{\vec{J}}(\vec{x},t_0) + \cdots\right)$$

Dipole Radiation

Lowest non $\Rightarrow$ $\phi = \frac{\int d^3x'\,\rho(\vec{x}',t_0)}{r} + \frac{\hat{r}}{r^2}\cdot\int \vec{x}'\,\rho(\vec{x}',t_0)\,d^3x' + \frac{\hat{r}}{cr}\int \vec{x}'\,\dot{\rho}(\vec{x}',t_0)\,d^3x'$

$$\boxed{\begin{gathered}\Rightarrow \phi = \underbrace{\frac{q}{r} + \frac{\hat{r}\cdot\vec{p}(t_0)}{r^2}}_{\text{quasi-static}} + \underbrace{\frac{\hat{r}}{cr}\cdot\dot{\vec{p}}(t_0)}_{\text{radiation and induction}} + \cdots \\ q = \int d^3x'\,\rho(\vec{x}',t_0) = \text{monopole moment (independent of time for localized source)} \\ \vec{p}(t_0) = \text{dipole moment at the time } t_0 = t - \frac{r}{c}\end{gathered}}$$

$$\vec{A} = \frac{1}{cr}\int d^3x'\,\vec{J}(\vec{x}',t_0) + \frac{\hat{r}}{cr^2}\cdot\int d^3x'\,\vec{x}'\,\vec{J}(\vec{x}',t_0)$$

$$+ \frac{\hat{r}}{c^2r}\cdot\int \vec{x}'\,\dot{\vec{J}}(\vec{x}',t_0)\,d^3x' + \cdots$$

Aside: $\int d^3x' \, \vec{J}(\vec{x}', t_0) = -\int d^3x \, \vec{x}' \underbrace{\vec{\nabla} \cdot \vec{J}(\vec{x}', t_0)}_{-\dot{\rho}} = \dot{\vec{p}}(t_0)$

To lowest non-vanishing term:

$$\boxed{\vec{A}(\vec{x}, t) = \frac{\dot{\vec{p}}(t_0)}{cr} + \cdots}$$

radiation and induction contribution

Now the Fields:

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \nabla\phi = -\frac{\ddot{\vec{p}}}{c^2 r} - \vec{\nabla}\phi$$

Aside: $\nabla\phi = \frac{q}{r^2}\hat{r} + \frac{3\hat{r}(\hat{r}\cdot\vec{p}(t_0)) - \vec{p}(t_0)}{r^3}$

$$- \frac{x_j}{r^3} \vec{\nabla} p_j(t_0) - \vec{\nabla}\left(\frac{\hat{r}}{cr} \cdot \dot{\vec{p}}(t_0)\right)$$

Double aside:

$$\vec{\nabla} p_j(t_0) = \dot{p}_j \vec{\nabla} t_0 = -\frac{\dot{p}_j}{c} \vec{\nabla} r = -\frac{\dot{p}_j}{c} \hat{r}$$

$$\Rightarrow \frac{x_j}{r^3} \vec{\nabla} p_j(t_0) = -(\hat{r} \cdot \dot{\vec{p}}(t_0)) \frac{\hat{r}}{cr^2}$$

Triple aside: $\vec{\nabla}\left(\frac{\hat{r}}{cr} \cdot \dot{\vec{p}}(t_0)\right) = \frac{\dot{\vec{p}}}{cr^2} - \frac{2(\hat{r}\cdot\dot{\vec{p}})\hat{r}}{cr^2}$

$$- \frac{(\ddot{\vec{p}}\cdot\hat{r})\hat{r}}{c^2 r}$$

$$\boxed{\begin{aligned}\vec{E} = {} & \frac{q}{r^2}\hat{r} + \frac{3\hat{r}(\hat{r}\cdot\vec{p}(t_0)) - \vec{p}(t_0)}{r^3} && \left(\begin{matrix}\text{Quastic static fields} \\ \text{Near field zone}\end{matrix}\right) \\ & + \frac{3\hat{r}(\hat{r}\cdot\dot{\vec{p}}(t_0)) - \dot{\vec{p}}(t_0)}{c r^2} && \left(\begin{matrix}\text{Induction field} \\ \text{Intermediate zone}\end{matrix}\right) \\ & - \frac{1}{c^2}\left(\frac{\ddot{\vec{p}}(t_0) - \hat{r}(\hat{r}\cdot\ddot{\vec{p}}(t_0))}{r}\right) && \left(\begin{matrix}\text{Radiation field} \\ \text{Far zone}\end{matrix}\right)\end{aligned}}$$

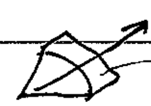
~~Radiation~~ $\vec{B} = \vec{\nabla}\times\vec{A} = \vec{\nabla}\times\left(\frac{\dot{\vec{p}}(t_0)}{cr}\right) = \frac{1}{cr}\left(\vec{\nabla}\times\dot{\vec{p}}(t_0)\right) - \frac{\dot{\vec{p}}(t_0)}{c}\times\vec{\nabla}\frac{1}{r}$

<u>Aside:</u> $(\vec{\nabla}\times\dot{\vec{p}}(t_0))_i = \epsilon_{ijk}\,\partial_j\,\dot{p}_k(t_0) = \epsilon_{ijk}\,\ddot{p}_k\,\partial_j(t_0)$

$$= -\epsilon_{ijk}\,\ddot{p}_k(t_0)\,\frac{\hat{r}_j}{c} = \ddot{\vec{p}}(t_0)\times\frac{\hat{r}}{c}$$

$$\Rightarrow \boxed{\vec{B}(\vec{x},t) = \underbrace{\frac{\dot{\vec{p}}(t_0)\times\hat{r}}{c r^2}}_{\text{Induction}} + \underbrace{\frac{\ddot{\vec{p}}(t_0)\times\hat{r}}{c^2 r}}_{\text{Radiation}}}$$

<u>Radiation fields fall off like $\frac{1}{r}$</u>

Why? Intensity $\underset{I}{\langle S\rangle} \sim |E|^2$  $da = r^2 d\Omega$

Energy flow to infinity $\int \langle S\rangle\cdot d\vec{a} \sim \int |E|^2 r\, d\Omega$

Finite as $r\to\infty \Rightarrow I \sim \frac{1}{r^2} \Rightarrow E \to \frac{1}{r}$

L# 18.5

Field Character different in different regions of space

Consider case $q_{net} = 0$

$$E \sim \frac{p}{r^3} + \frac{\dot{p}}{r^2 c} + \frac{\ddot{p}}{c^2 r} \qquad \dot{p} \sim \frac{p}{T}, \quad \ddot{p} \sim \frac{p}{T^2}$$

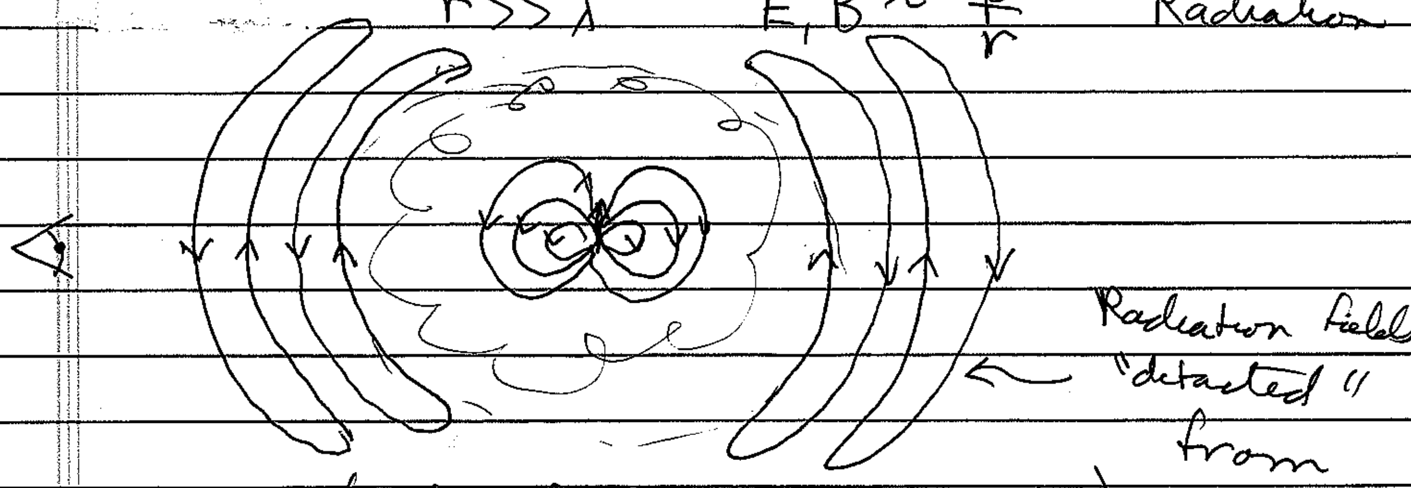
$$\Rightarrow E \sim \frac{p}{r^3}\left(1 + \frac{r}{cT} + \left(\frac{r}{cT}\right)^2\right), \qquad T \sim \frac{1}{\omega} \Rightarrow cT \sim \lambda$$

$$\Rightarrow E \sim \frac{p}{r^3}\left(1 + \frac{r}{\lambda} + \left(\frac{r}{\lambda}\right)^2\right)$$

Near field $r \ll \lambda$ $E \sim \frac{p}{r^3}$, $B = 0$ ~~Quasi~~ Quasi-static

Intermediate $r \sim \lambda$ $E, B \sim \frac{\dot{p}}{r^2}$ Induction

$r \gg \lambda$ $E, B \sim \frac{\ddot{p}}{r}$ Radiation



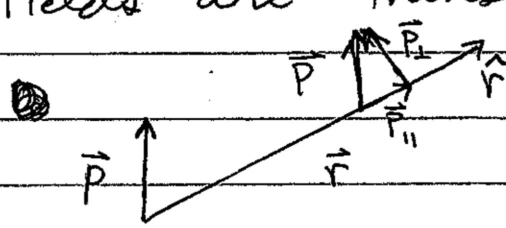
(Electric field at ~~a~~ some instant)

Radiation fields propagate energy to infinity. Unlike quasi-static case energy cannot be ~~recovered~~ brought back to the charge configuration

Electromagnetic dipole radiation

$$\boxed{\vec{E}_{rad} = -\left(\frac{\ddot{\vec{p}}(t_0) - \hat{r}(\ddot{\vec{p}}(t_0)\cdot\hat{r})}{c^2 r}\right) \qquad \vec{B}_{rad} = \frac{\ddot{\vec{p}}(t_0)\times\hat{r}}{c^2 r}}$$

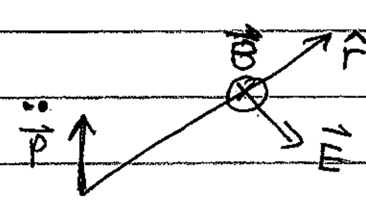
Fields are "transverse" to observation direction



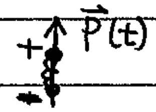
$$\vec{p}_{\parallel} = \hat{r}\cdot\vec{p}$$

$$\vec{p}_\perp = \vec{p} - \vec{p}_{\parallel} = \hat{r}\times(\vec{p}\times\vec{r})$$

$$\Rightarrow \boxed{\vec{E}_{rad} = -\frac{\ddot{\vec{p}}_\perp(t-\frac{r}{c})}{c^2 r} \qquad \vec{B}_{rad} = \hat{r}\times\vec{E}_{rad}}$$



Example: Linearly oscillating dipole: $\vec{p} = p_0 \cos\omega t\, \vec{e}_z$



$$\vec{p}(t) = \vec{p}_0 \cos\omega t = \mathrm{Re}(\vec{p}_0 e^{-i\omega t})$$

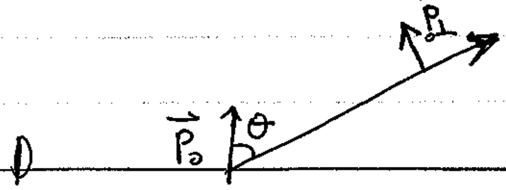
Look a complex fields. Take real part in the end

$$\vec{E}_{rad} = -\frac{\ddot{\vec{p}}_\perp(t-\frac{r}{c})}{c^2 r} = \frac{\omega^2}{c^2}\vec{p}_{0\perp}\frac{e^{-i\omega(t-\frac{r}{c})}}{r}$$

$$\Rightarrow \vec{E}_{rad} = k^2 \vec{p}_{0\perp}\frac{e^{i(kr-\omega t)}}{r}$$

spherical wave of amplitude $\vec{p}_{0\perp}$

L#18.7



$$\vec{p}_{0\perp} = \hat{r} \times (\vec{p}_0 \times \hat{r}) = -p_0 \sin\theta\, \hat{\theta}$$

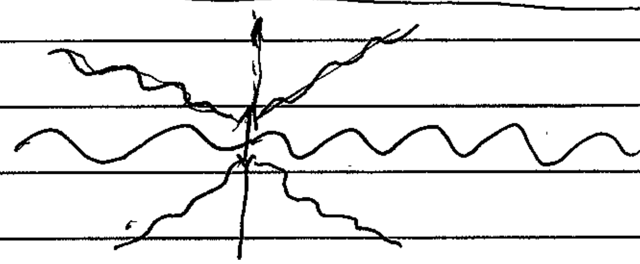
$$\vec{p}_0 \times \hat{r} = p_0 (\hat{z} \times \hat{r}) = p_0 \sin\theta\, \hat{\phi}$$

$$\Rightarrow \boxed{\begin{aligned} \vec{E}_{rad}(\vec{x},t) &= -k^2 \sin\theta\, p_0\, \frac{e^{i(kr-\omega t)}}{r}\, \hat{\theta} \\ \vec{B}_{rad}(\vec{x},t) &= \frac{\ddot{\vec{p}} \times \hat{r}}{c^2 r} = -k^2 \sin\theta\, p_0\, \frac{e^{i(kr-\omega t)}}{r}\, \hat{\phi} \end{aligned}}$$

Anisotropic transverse vector spherical wave

$|\vec{E}_{rad}| = |\vec{B}_{rad}| \qquad \vec{E}_{rad} \perp \vec{B}_{rad} \perp \vec{k} \qquad \vec{B} = \hat{k} \times \vec{E}$

$\vec{k} = k\hat{r}$

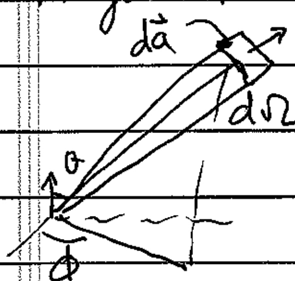


Think of anisotropy like waves on a string

Energy flux: $\langle \vec{S} \rangle = \frac{c}{8\pi} \vec{E}^*_{rad} \times \vec{B}_{rad} = \frac{c}{8\pi} E_{rad}^2\, \hat{r}$

$$\langle \vec{S} \rangle = \frac{ck^4}{8\pi}\, p_0^2\, \frac{\sin^2\theta}{r^2}\, \hat{r} \propto \frac{1}{r^2}$$

Angular distribution of radiated power



$d\vec{a} = \hat{r}\, r^2 d\Omega \Rightarrow dP_{ower} = \langle \vec{S} \rangle \cdot d\vec{a}$

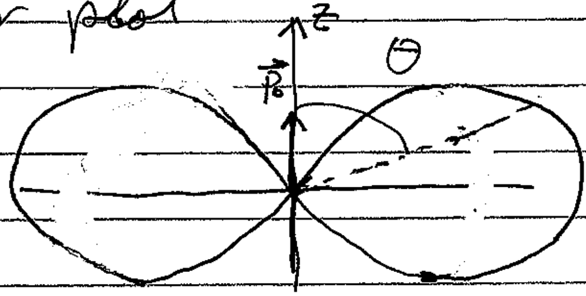
Power radiated per unit solid angle

$$\frac{dPower}{d\Omega} = \langle \vec{S} \rangle \cdot r^2 \hat{r}$$

For linear oscillating dipole:

$$\boxed{\frac{d\text{Power}}{d\Omega} = \frac{c k^4}{8\pi} p_0^2 \sin^2\theta}$$

Polar plot



Dipole radiation pattern (independent of ϕ)

Total power radiated into all directions

$$\text{Power} = \int \frac{dP}{d\Omega} d\Omega = \frac{c k^4}{8\pi} p_0^2 \underbrace{\int \sin^2\theta \, d\Omega}_{\to \frac{8\pi}{3}}$$

$$\Rightarrow \boxed{\text{Power} = \frac{c k^4}{3} p_0^2}$$ Larmor's formula

For a single moving charge

$$\vec{p}(t) = q\,\vec{x}(t) \quad \Rightarrow \quad \vec{E}_{rad} = \frac{-q\,\ddot{\vec{x}}_\perp(t - \frac{r}{c})}{c^2 r}$$

$$\Rightarrow \boxed{\vec{E}_{rad} = -\frac{q}{c^2 r} \vec{a}_\perp(t - \tfrac{r}{c})}$$

Radiated field proportional to acceleration of charge

Instaneous total power radiated

$$\boxed{\text{Power}(t) = \frac{2}{3}\frac{q^2}{c^3} \left|\vec{a}(t - \tfrac{r}{c})\right|^2}$$