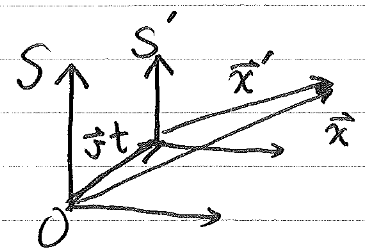
Lecture #22 : Special Relativity Review

I: History

Galileo: Principle of inertia $\equiv$ body in uniform motion stays in motion

$\Rightarrow$ no special "rest frame" $\equiv$ all "inertial frames" equiv.



Reference frame S' moves with velocity $\vec{v}$ w.r.t. S (take origins equal @ $t=0$)

$$\vec{x}' = \vec{x} + \vec{v}\,t \quad , \quad t' = t \text{ (intuitive)}$$

Principle of relativity: Physical laws are equivalent in all inertial frames unless frame choice explicitly appears

- E.g. ~~Physical~~ Newton's Gravitation

$$m_i \frac{d^2\vec{r}_i}{dt^2} = -G \sum_{i,j} m_i m_j \frac{\vec{r}_i - \vec{r}_j}{|\vec{r}_i - \vec{r}_j|^3}$$

Galilean transf. $\Rightarrow$

$$m_i \frac{d^2\vec{r}_i{}'}{dt^2} = -G \sum_{i,j} m_i m_j \frac{\vec{r}_i{}' - \vec{r}_j{}'}{|\vec{r}_i{}' - \vec{r}_j{}'|^3}$$

Manifestly Covariant

- E.g.: Wave propagation (e.g. water)

$$\left(\nabla^2 - \frac{1}{w_p^2}\frac{\partial^2}{\partial t^2}\right)\psi(\vec{x},t) = 0$$

Galilean transf:

$$\left(\nabla'^2\left(1 - \frac{v^2}{w_p^2}\right) - 2\frac{\vec{v}\cdot}{w^2}\vec{\nabla}'\frac{\partial}{\partial t'} - \frac{1}{w^2}\frac{\partial'}{\partial t'^2}\right)\psi(\vec{x}',t') = 0$$

Not manifestly covariant, but not expected since w is the phase velocity in the rest frame of medium

What about electromagnetic waves? In what frame is the speed of light c?

- Maxwell $\Rightarrow$ Luminiferous Ether (incompressible fluid ~~solvent~~ surrounding us)

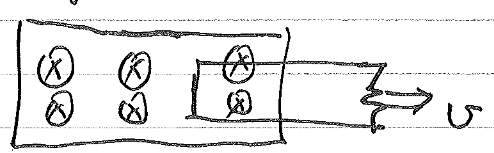
- 1897 : Michelson Morley Experiment (null result)

  - Michelson (Ether is "dragged")
  - Fitzgerald (Length contracts)
  - Lorentz (electron contracts, defines Lorentz transf.)

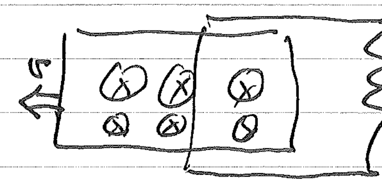
- 1905 : Einstein's theory of special relativity

Not really aware of Michelson-Morley, more concerned with the inherent assymmetries

Faraday's "Universal law" of Induction



vs.



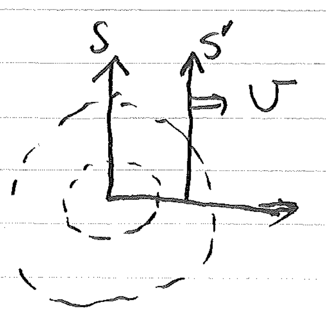
Motional EMF: $\vec{F} = q\vec{v}\times\vec{B}$

Changing $\vec{B} \Rightarrow$ induced EMF

Coincidence?

II Principle of Relativity a'la Einstein

(1) Laws of physics equivalent in all inertial ref. frames
(2) Speed of light is the same for all observers



If at $t=t'=0$ a spherical pulse is emitted from origin

$$x^2+y^2+z^2 = c^2t^2$$
$$x'^2+y'^2+z'^2 = c^2t'^2$$

Time must be different in different frames!

Derivation of the Lorentz Transformations

Linear transformation (homogenity of space) ~~with~~ which goes to Galilean transformation when $c \to \infty$

Relative motion along x : $x' = a(v)\,(x - vt)$ $\qquad t' = b(v)\,(t - d(v)\,x)$

Require $a \to 1$, $b \to 1$, $d \to 0$ as $\frac{v}{c} \to 0$

(Use inverse $v \to -v$): $\quad x = a(-v)\,(x' + vt)$
$\qquad t = b(-v)\,(t' - d(-v)\,t)$

$$\Rightarrow a(v) = b(v) = \frac{1}{\sqrt{1 - v\,d(v)}}$$

Invariant interval: $x'^2 = c^2 t'^2$, $\quad x^2 = c^2 t^2 \quad (y = z = 0)$

$$\Rightarrow a(v) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \equiv \gamma, \quad d(v) = \frac{v}{c}$$

Lorentz tranformation with S' at $v\vec{e}_x$ w.r.t S

$$\boxed{\begin{array}{ll} x' = \gamma(x - vt) & x = \gamma(x' + vt') \\ t' = \gamma\left(t - \frac{v}{c^2} x\right) & t = \gamma\left(t' + \frac{v}{c^2} x'\right) \\ \multicolumn{2}{c}{y = y', \quad z = z'} \\ \multicolumn{2}{c}{\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}, \quad \beta \equiv \frac{v}{c}} \end{array}}$$

New geometry: Space - Time

Four coordinates: $(x^0, \vec{x})$ $\qquad x^0 = ct \quad x^1 = x,\ x^2 = y,\ x^3 = z$

$$x^{0\prime} = \gamma(x^0 - \beta x^1)$$

$$x^{1\prime} = \gamma(x^1 - \beta x^0)$$

(like a rotation with respect to an imaginary angle)

Consequences of relativity

- Velocity addition (same direction)

$$u = \frac{dx}{dt} \Rightarrow u' = \frac{\gamma(dx - v\,dt)}{\gamma(dt - \frac{v}{c^2}dx)} =$$

$$\Rightarrow \boxed{u' = \frac{u - v}{1 - \frac{uv}{c^2}}} \qquad u = c$$

$$\Rightarrow u' = \frac{c - v}{1 - v/c} = c$$

c : Maximum speed

- Simultaneity: Event A: $(x_A, t_A = t)$; Event B $(x_B, t_B = t)$ (S)

In co-moving frame: $\Delta t' = t_B' - t_A' = \gamma \frac{v}{c^2}(x_B - x_A) \neq 0$!

(Virtually every paradox of relativity is a question of simultaneity)

- Time dilation: Proper (own) time = time interval between two events which happen at the same space point (time as measured by a watch in its own rest frame)

Event A: (x, t_A) : Event B (x, t_B)

$\Rightarrow$ Proper time $dt = t_B - t_A \equiv d\tau$

$$\boxed{dt' = t_B' - t_A' = \gamma\, d\tau}$$ Moving clocks tick slower!

- Length contraction

$\Delta L|_{proper}$ = distance between ends of the rod at the same time

ruler moving $\longrightarrow v$

$$\begin{pmatrix} t_A = 0, & x_A = 0 \\ t_B = 0 & x_B = \Delta L_{lab} \end{pmatrix}$$

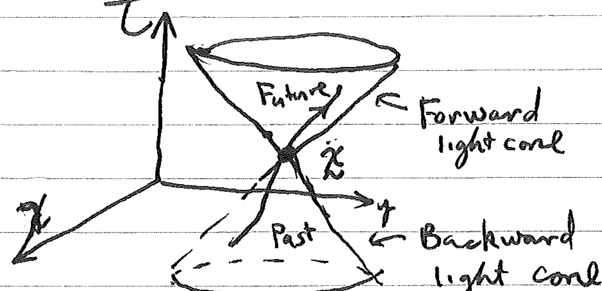
In ruler's rest frame

$$\begin{pmatrix} t_A' = 0 & x_A' = 0 \\ t_B' = \gamma \frac{v}{c^2}\Delta L|_{Lab} & x_B' = \gamma \Delta L_{lab} \end{pmatrix}$$

$$\Rightarrow \Delta L_{proper} = x_B' - x_A' = \gamma \Delta L_{lab} \Rightarrow \boxed{\Delta L_{lab} = \frac{\Delta L_{prop}}{\gamma}}$$

Geometry of Space-time Diagrams

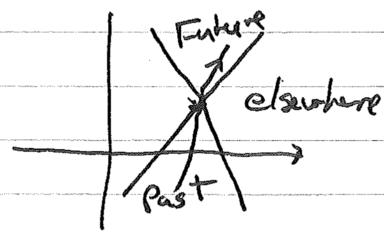
Event ≡ geometric point in space-time
Coordinates (x^0, x^1, x^2, x^3) : Minkowski 4-Vector



World line – trajectory of a particle in space-time

Light cone – Surface of points traced out by a spherical light pulse eminating from (or converging on) a space-time point

Given two events: $|\Delta\vec{x}|^2 < (ct)^2$ Causually connected



$|\Delta\vec{x}|^2 > (ct)^2$ Not causually connected

$|\Delta\vec{x}|^2 = (ct)^2$ Connected by a light signal

- Invariant Interval $\boxed{c^2\Delta t^2 - |\Delta\vec{x}|^2 = \Delta S^2}$ Same in all reference frames

$\Delta S^2 \equiv$ effective length in Minkowski space

- $\Delta S^2 = 0 \Rightarrow$ Events are light-like separated
- $\Delta S^2 < 0 \Rightarrow$ Events are space-like separated

  i.e. $\exists$ a frame in which $\Delta t = 0 \Rightarrow \Delta S^2 = |\Delta\vec{x}|^2$

  (Not causally connected)
- $\Delta S^2 > 0 \Rightarrow$ Events are time-like separated

  $\Rightarrow \exists$ a frame s.t. $\Delta x = 0 \Rightarrow \Delta t = \Delta\tau$ proper time

  $\Rightarrow \Delta S^2 = c^2 (\Delta\tau)^2$

Note: Given Δt for two events given particle moving with velocity u, $\Delta\tau = \frac{\Delta t}{\gamma(u)}$ is an invariant

Geometry of space time

- **Recall 3D Euclidean geometry**

Position vector $\vec{x} = (x^1, x^2, x^3)$

Under rotation to a new coordinate system:

$$x_i' = R_{ij}\, x_j \quad : \left(\text{Rotation matrix } R_{ij}\right)$$

(Einstein summation convention)

Length is preserved $x_i' x_i' = x_j x_j = |\vec{x}|^2$

Define a vector : transforms like position $V_i' = R_{ij} V_j$

- **In Minkowski space**

Position 4-vector $(x^0, x^1, x^2, x^3) = x^\mu$ (Contravariant coordinates)

Under Lorentz tranformation:

$$x'^\mu = \Lambda^\mu{}_\nu\, x^\nu$$

(Summation convention, repeated index, one lower one upper)

$$\begin{aligned} x'^0 &= \gamma(x^0 - \beta x^1) \\ x'^1 &= \gamma(x^1 - \beta x^0) \\ x'^2 &= x^2 \\ x'^3 &= x^3 \end{aligned}$$

("Boost" along x^1)

Define a general Four-vector (contravariant) $V^\mu = (V^0, V^i_{\,i=1,2,3})$

Transforms like position

$$\boxed{V^{\mu\prime} = \Lambda^\mu{}_\nu V^\nu}$$

Lorentz Scalar field: $\psi'(x') = \psi(x)$ No change in value under Lorentz Transf

Metric

Under Lorentz Transformation the "distance" is preserved

$\Rightarrow$ Inner product $x \cdot x = (x^0)^2 - ((x^1)^2 + (x^2)^2 + (x^3)^2)$

$$= (x^0)^2 - |\vec{x}|^2 = x' \cdot x'$$

Define metric tensor $x^\mu g_{\mu\nu} x^\nu = x \cdot x$

$$\Rightarrow \quad g_{\mu\nu} = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$$

Defines the distance in Minkowski space

Define covariant coordinates $x_\mu = g_{\mu\nu} x^\nu = (x_0, -\vec{x})$

(Geometrically the coordinates w.r.t. to the dual basis)

Generally for any two four vectors

$V^\mu W_\mu$ = Lorentz scalar $\quad W_\mu = g_{\mu\nu} W^\nu$

Also $g^{\mu\nu} = g_{\mu\nu} = g_{\mu\nu}^{-1} \Rightarrow W^\nu = g^{\nu\mu} W_\mu$ etc.

Example: Relativistic Doppler Shift

Consider a plane wave with wave vector $\vec{k}$, frequency ω

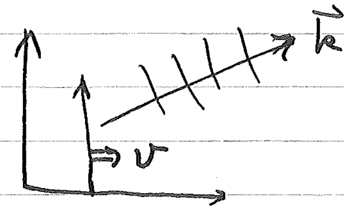
Define phase field $\phi(x) = \vec{k} \cdot \vec{x} - \omega t$

Physically, this must be a Lorentz scalar field

(Wave reaches its peak at a certain space-time point)

$$\Rightarrow \phi'(x') = \vec{k}' \cdot \vec{x}' - \omega' t' = \phi(x)$$

$$\Rightarrow \phi(x) = \vec{k}\cdot\vec{x} - \omega t = k_\parallel x_\parallel + \vec{k}_\perp\cdot\vec{x}_\perp - \frac{\omega}{c} x^0$$



, parallel component along $\vec{v}$

, $\perp$ to v: $\vec{x}_\perp$

$$x_\parallel = \gamma(x'_\parallel + \beta x^{0\prime}) \qquad x^0 = \gamma(x^{0\prime} + \beta x'_\parallel)$$

$$x_\perp = x'_\perp$$

$$\Rightarrow \phi(x) = k_\parallel\left(\gamma(x'_\parallel + \beta x^{0\prime})\right) + \vec{k}_\perp\cdot\vec{x}_\perp - \frac{\omega}{c}\left(\gamma(x^{0\prime} + \beta x'_\parallel)\right)$$

$$= \gamma\left(k_\parallel - \beta\frac{\omega}{c}\right)x'_\parallel + \vec{k}_\perp\cdot\vec{x}'_\perp - \gamma\left(\frac{\omega}{c} - \beta k_\parallel\right)x'_0$$

$$= \phi(x') = k'_\parallel x'_\parallel + \vec{k}'_\perp\cdot\vec{x}'_\perp - \frac{\omega'}{c}x_0$$

Thus $\boxed{k^\mu = (k^0, \vec{k}) \text{ with } k^0 = \frac{\omega}{c} \text{ is a four vector}}$

$$k^{0\prime} = \gamma(k^0 - \beta k_\parallel) \Rightarrow \boxed{\omega' = \gamma(\omega - v k_\parallel)}$$

$$k'_\parallel = \gamma(k_\parallel - \beta k_0) \Rightarrow \boxed{k'_\parallel = \gamma\left(k_\parallel - \frac{v}{c^2}\omega\right)}$$

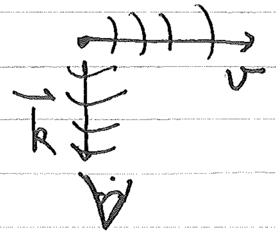
$$\vec{k}'_\perp = \vec{k}_\perp$$

$$\beta \ll 1 \quad \gamma \sim 1 \quad \omega' = \omega - k_\parallel v$$

<u>Note</u>: Even for ~~waves~~ emitter moving $\perp$ to observer there is a Doppler shift in ω

$k_\parallel = 0$

$|\vec{k}| = \frac{\omega}{c}$



$$\omega' = \gamma\omega \approx \left(1 - \frac{v^2}{2c^2}\right)\omega$$

quadratic

Generally $\tan\theta = \frac{|\vec{k}_\perp|}{k_\parallel}$ changes

Other four vectors

- *Proper velocity*: First note that $\frac{dx^\mu}{dt}$ is *not* a four-vector since t transforms under Lorentz.

However if we use the proper time (i.e. time in the particles own rest frame): $d\tau$

$$u^\nu = \frac{dx^\nu}{d\tau}, \quad \text{Now } d\tau = \frac{dt}{\gamma(u)}$$ where u = speed in Lab frame

$$\Rightarrow \boxed{u^\nu = \gamma(u)\frac{dx^\nu}{dt} = (\gamma(u)c, \gamma(u)\vec{u})}$$

- ~~Four momentum~~ Relativistic invariant

$$u^\nu u_\nu = (\gamma(u))^2 c^2 - \gamma(u)^2 u^2 = \gamma(u)^2(c^2-u^2)$$
$$= \frac{1}{1-\frac{u^2}{c^2}}(c^2-u^2) = c^2 \checkmark$$

- ~~Proper~~ *Four-momentum*

$$p^\nu = \underset{\substack{\uparrow \\ \text{rest mass}}}{m} u^\nu = \underset{\substack{\uparrow \\ \text{relativistic mass}}}{\gamma(u)m}\frac{dx^\nu}{dt} = (\gamma(u)mc, \gamma(u)m\vec{u})$$

Empirical ~~fact~~: All components of p^ν are *conserved*

$$p^0 = \gamma(u)mc = E/c \quad \text{relativistic energy}/c$$

Limit $u \ll c$ $\quad \gamma(u) \approx 1 + \frac{u^2}{2c^2} \Rightarrow cp^0 = \underset{\substack{\uparrow \\ \text{rest energy}}}{mc^2} + \frac{1}{2}mu^2$

Relativistic invariant $\quad p^\nu p_\nu = (mc)^2 = \left(\frac{E}{c}\right)^2 - |\vec{p}|^2$

- **Four-current density**

Charge conservation: True in all reference frames

$$\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0 \Rightarrow \vec{\nabla}'\cdot\vec{J}' + \frac{\partial \rho'}{\partial t'} = 0$$

$$\parallel$$

$$\frac{\partial}{\partial x_{\parallel}} J_{\parallel} + \vec{\nabla}_{\perp}\cdot\vec{J}_{\perp} + \frac{\partial \rho}{\partial t} = 0$$

Chain rule ~~Where~~:

$$\begin{cases} \dfrac{\partial}{\partial x_{\parallel}} = \dfrac{\partial x'_{\parallel}}{\partial x_{\parallel}}\dfrac{\partial}{\partial x'_{\parallel}} + \dfrac{\partial t'}{\partial x_{\parallel}}\dfrac{\partial}{\partial t'} = \gamma\left(\dfrac{\partial}{\partial x'_{\parallel}} - \dfrac{v}{c^2}\dfrac{\partial}{\partial t'}\right) \\ \dfrac{\partial}{\partial t} = \gamma\left(\dfrac{\partial}{\partial t'} - v\dfrac{\partial}{\partial x'_{\parallel}}\right) \\ \vec{\nabla}'_{\perp} = \vec{\nabla}_{\perp} \end{cases}$$

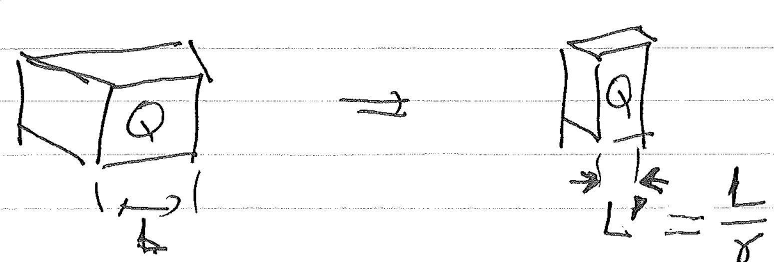
$$\Rightarrow \vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = \gamma\left(\frac{\partial}{\partial x'_{\parallel}} - \frac{v}{c^2}\frac{\partial}{\partial t'}\right)J_{\parallel} + \vec{\nabla}'_{\perp}\cdot\vec{J}_{\perp} + \gamma\left(\frac{\partial}{\partial t'} - v\frac{\partial}{\partial x'_{\parallel}}\right)\rho$$

$$= \frac{\partial}{\partial x'_{\parallel}}\left[\gamma(J_{\parallel} - v\rho)\right] + \vec{\nabla}'_{\perp}\cdot\vec{J} + \frac{\partial}{\partial t'}\left[\gamma\left(\rho - \frac{v}{c^2}J_{\parallel}\right)\right]$$

$$\therefore \boxed{J^{\mu} = (c\rho, \vec{J}) \text{ is a Four vector}}$$

$\vec{J} = 0$ in S frame $\Rightarrow$ $\rho' = \gamma\rho$ is S'

Physically: Length contraction



Length contraction

<u>Minkowski Force:</u> $K^\mu = \dfrac{dp^\mu}{d\tau} = m\dfrac{d^2x^\mu}{d\tau^2}$

"Kinematic force" $\vec F = \dfrac{d\vec p}{dt}$, $K^\mu = \gamma \dfrac{dp^\mu}{dt}$

$\Rightarrow \quad K^\mu = \left(\dfrac{\gamma}{c}\dfrac{dE}{dt},\ \gamma\vec F\right)$

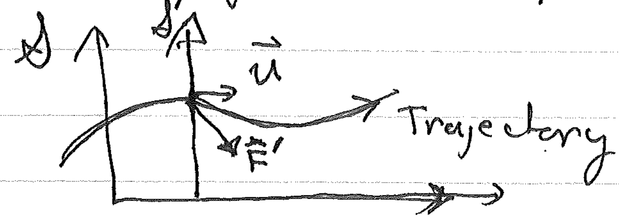
Aside $\quad u^\mu u_\mu = c^2 \quad \Rightarrow \quad \dfrac{du^\mu}{d\tau} u_\mu = 0$

$\Rightarrow \quad K^\mu u_\mu = 0 = \gamma^2 \dfrac{dE}{dt} - \gamma^2 \vec F\cdot\vec u$

$\therefore \quad \dfrac{dE}{dt} = \vec F\cdot\vec u$ (rate at which force does work on particle)

$$\Rightarrow \boxed{K^\mu = \left(\gamma \vec F\cdot\frac{\vec u}{c},\ \gamma\vec F\right)}$$

Suppose we boost to the instantaneous rest frame of the particle



Force in particle frame

$K'^\mu = (0, \vec F')$

since $\vec u = 0$ in frame S'

In Lab frame: $K^\mu \left(\gamma \vec F\cdot\dfrac{\vec u}{c},\ \gamma\vec F\right)$

If Boost along $\vec F$: $\vec K_\parallel = \gamma(\vec K'_\parallel + u \underbrace{K'^0}_{0} \hat e_\parallel)$

$\Rightarrow \vec K_\parallel = \gamma \vec F'_\parallel = \gamma \vec F_\parallel$

$$\Rightarrow \boxed{\vec{F}_{\parallel} = \vec{F}_{\parallel}^{\,p}}$$

If Boost $\perp$ to $\vec{F}$

$$\Rightarrow \vec{K}_{\perp} = \vec{K}_{\perp}^{\,p} \Rightarrow \boxed{\gamma(\vec{u})\,\vec{F}_{\perp} = \vec{F}_{\perp}^{\,p}}$$

These are the transformation rules on forces where $\vec{F}^{\,p}$ is the instaneous force on the particle in its own rest frame.