Lecture 22: Special Relativity Review

I. History

Galileo: Principle of inertia: body in uniform motion stays in motion.

⇒ No special "rest frame" = all "inertial frames" are equivalent.

Reference frame $S'$ moves with velocity $\vec{v}$ w.r.t. $S$ (take origin at equal $\theta^t = 0$)

\[
\vec{x}' = \vec{x} + \vec{v} \cdot t' \quad \text{and} \quad t' = t \quad \text{(in relative)}
\]

Principle of relativity: Physical laws are equivalent in all inertial frames unless frame change explicitly appears.

- Eg. Minkowski: Newton's Gravitation

\[
m_1 \frac{d^2 \vec{r}_i}{dt^2} = - G \sum_{i,j} m_i m_j \frac{\vec{r}_i - \vec{r}_j}{r^3}
\]

Galilean: \[ m_i \frac{d^2 \vec{r}_i}{dt^2} = - G \sum_{i,j} m_i m_j \frac{\vec{r}_i - \vec{r}_j}{r^3} \]

Mach's Principle: covariant

- Eg. Wave propagation (e.g. water)

\[
\left( \nabla^2 - \frac{1}{w^2} \frac{\partial^2}{\partial t^2} \right) \psi(x,t) = 0
\]

Galilean:

\[
\left( \nabla^2 - \frac{\partial^2}{\partial t^2} \right) \psi(x,t) = 0
\]

Galilean:

\[
\left( \nabla^2 - \frac{v^2}{w^2} \right) - 2 \frac{\vec{v} \cdot \nabla}{w^2} \frac{\partial}{\partial t'} - \frac{1}{w^2} \frac{\partial^2}{\partial t'^2} \right) \psi(x',t') = 0
\]

Not manifestly covariant, but not expected since $w$ is the phase velocity in the rest frame of medium.
What about electromagnetic waves? In what frame is the speed of light $c$?

- Maxwell $\Rightarrow$ Luminiferous Ether (incompressible fluid surrounding us)

- 1897: Michelson Morley Experiment (null results)
  - Michelson (Ether is "dragged")
  - Fitzgerald (Length contracts)
  - Lorentz (electron contracts, defines Lorentz transform)

- 1905: Einstein's theory of special relativity
  Not really aware of Michelson-Morley, more concerned with the inherent asymmetries

Faraday's "Universal law" of Induction

![Diagram of Faraday's law](image)

Motional EMF: $\mathbf{E} = q \mathbf{v} \times \mathbf{B}$

Changing $\mathbf{B}$ $\Rightarrow$ Induced EMF

Coincidence?

II Principle of Relativity a la Einstein

1. Laws of physics equivalent in all inertial ref. frames
2. Speed of light is the same for all observers

If at $t = t' = 0$ a spherical pulse is emitted from origin

- $x^2 + y^2 + z^2 = c^2 t^2$ (time must be different in different frames)
Derivation of the Lorentz Transformations

Linear transformation (homogeneity of space) which goes to Galilean transformation when \( c \rightarrow 0 \)

Relative motion along \( x \):
\[
x' = a(v) (x - vt) \quad t' = b(v) (t - dv) x
\]

Assume \( a \rightarrow 1 , b \rightarrow 1 , d \rightarrow 0 \) as \( \frac{v}{c} \rightarrow 0 \)

\begin{align*}
\text{(Use inverse): } \quad x &= a(-v) (x' + vt) \\
\quad t &= b(-v) (t' - d(-v) t)
\end{align*}

\[ a(v) = b(v) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \]

Invariant interval:
\[ x'^2 = c^2 t'^2 , \quad x^2 = c^2 t^2 \quad (y = z = 0) \]

\[ a(v) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma , \quad d(v) = \frac{v}{c} \]

Lorentz transformation with \( s \) at \( \mathbf{v} \) w.r.t. \( s \)

\[
\begin{align*}
x' &= \gamma (x - vt) \\
\quad t' &= \gamma (t - \frac{v}{c^2} x)
\end{align*}

\[ y = y', \quad z = z' \]

\[ \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} , \quad \beta = \frac{v}{c} \]

New geometry: Space-Time

Four coordinates: \((x^0, x)\)
\[ \begin{align*}
\bar{x} &= ct \quad x^0 = x , \quad \bar{x}^0 = \bar{y} , \quad \bar{x}^2 = z^2
\end{align*} \]

\[
\begin{align*}
x'^0 &= \gamma (x^0 - \beta x^1) \\
\quad x'^0 &= \gamma (x^1 - \beta x^0)
\end{align*}

(like a notation with respect to an imaginary angle)
Consequences of relativity

- **Velocity addition (same direction)**

\[
U = \frac{dx}{dt} \implies U' = \frac{y(dx - u dt)}{\gamma (dt - \frac{v}{c^2} dx)} = \frac{U - \frac{Uv}{c^2}}{1 - \frac{Uv}{c^2}} \implies U' = \frac{c - \frac{Uv}{c^2}}{1 - \frac{Uv}{c^2}} = c
\]

\[c: \text{ Maximum speed}\]

- **Simultaneity**: Event A: \((x_A, t_A = t)\); Event B \((x_B, t_B = t)\)

In co-moving frame: \(\Delta t' = t_B' - t_A' = \gamma \left(\frac{t_B - t_A}{c^2} (x_B - x_A) \right) \neq 0\)

(Virtually every paradox of relativity is a question of simultaneity)

- **Time dilation**: Proper (own) time = time interval between two events which happen at the same space point (time as measured by a watch in its own rest frame)

\[\text{Event A: } (x_A, t_A) \quad \text{Event B: } (x_B, t_B)\]

\[\implies \text{ Proper time } \Delta t' = t_B' - t_A' = \gamma \Delta t \]

\[\Delta t' = t_B' - t_A' = \gamma \Delta t \]

- **Length contraction**

\[\Delta L_{\text{proper}} = \text{distance between ends of the rod at the same time} \]

\[
\begin{align*}
(t_A = 0, & \quad x_A = 0) \\
(t_B = 0, & \quad x_B = \Delta L_{\text{lab}})
\end{align*}
\]

\[\Delta L_{\text{lab}} = \frac{\Delta L_{\text{proper}}}{\gamma} \]

\[\Delta L'_{\text{proper}} = x_B' - x_A' = \gamma \Delta L_{\text{lab}} \]
Geometry of Space-time Diagrams

Event = geometric point in space-time coordinates \((x^0, x^1, x^2, x^3)\): Minkowski 4-Vector

- World line - trajectory of a particle in space-time
- Light cone - Surface of points traced out by a spherical light pulse emanating from (or converging on) a space-time point

Given two events: \(\Delta x^2 < (ct)^2\) Causally connected
\(\Delta x^2 > (ct)^2\) Not causally connected
\(\Delta x^2 = (ct)^2\) Connected by a light signal

Invariant Interval \[c^2\Delta t^2 - \Delta x^2 = \Delta S^2\] Same in all reference frames
\(\Delta S^2 = \) effective length in Minkowski space

- \(\Delta S^2 = 0\) \(\Rightarrow\) Events are light-like separated
- \(\Delta S^2 < 0\) \(\Rightarrow\) Events are space-like separated
- \(\Delta S^2 > 0\) \(\Rightarrow\) Events are time-like separated

\(\Rightarrow\) In a frame s.t. \(\Delta x = 0\) \(\Rightarrow\) proper time
\(\Delta S^2 = c^2(\Delta \tau)^2\)

**Note:** Given \(\Delta t\) for two events, given particle moving with velocity \(v\), \(\frac{\Delta t}{\Delta \tau} = \frac{\Delta t}{\Delta x}\) is an invariant
Geometry of space-time

- Recall 3D Euclidean geometry.
  Position vector \( \mathbf{x} = (x^1, x^2, x^3) \)

  Under rotation to a new coordinate system:
  \[
  x'^i = \sum_j R_{ij} x^j \\
  \text{(Rotation matrix } R_{ij})
  \]
  (Einstein summation convention)

  Length is preserved \( x^a x_\alpha = x_j x_j = 1 \times 1^2 \)

  Define a vector: transforms like position \( V^\mu = R_{\mu j} V_j \)

- In Minkowski space

  Position 4-vector \((x^0, x^1, x^2, x^3) = x^\mu\) \( \text{(Contravariant)} \)

  Under Lorentz transformation:
  \[
  x'^\mu = \Lambda^\mu_\nu x^\nu \\
  \text{(Summation convention, repeated index, one lower one upper)}
  \]

  \[
  \begin{align*}
  x'^0 &= \gamma (x^0 - \beta x^1) \\
  x'^1 &= \gamma (x^1 - \beta x^0) \\
  x'^2 &= x^2 \\
  x'^3 &= x^3
  \end{align*}
  \]
  \( \gamma = \frac{1}{\sqrt{1 - \beta^2}} \)

  "Boost" along \( x^1 \)

  Define a general four-vector \( V^\mu = (V^0, V^i) \) \( \text{(Contravariant)} \)

  Transforms like position
  \[
  V'^\mu = \Lambda^\mu_\nu V^\nu
  \]

  Scaler: \( \phi' = \phi \) \( \text{No change in value under Lorentz Transform} \)
Metric

Under Lorentz Transformation the "distance" is preserved

\[ \Rightarrow \text{Inner product } \mathbf{x} \cdot \mathbf{x} = (x^0)^2 - (x^1)^2 - (x^2)^2 = (x^0)^2 - |\mathbf{x}|^2 = x^0 x^0 \]

Define metric tensor \( g^{\mu \nu} \)

\[ g^{\mu \nu} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \]

Defines the distance in Minkowski space

Define covariant coordinate \( X^\mu = g^{\mu \nu} X^\nu = (x^0, -\mathbf{x}) \)

(Geometrically the coordinate w.r.t. the dual basis)

Generally for any two four vectors \( V^\mu W_\mu = \text{Lorentz scalar } W_\mu g^{\mu \nu} W^\nu \)

Also \( g^{\mu \nu} = g_{\mu \nu} = g^{-1} \Rightarrow W^\nu = g^{\nu \mu} W_\mu \) etc.

Example: Relativistic Doppler Shift

Consider a plane wave with wave vector \( k \), frequency \( \omega \)

Define phase field \( \phi(x) = k \cdot x - \omega t \)

Physically, this must be a Lorentz scalar field

(Wave reaches its peak at a certain space-time point)

\[ \Rightarrow \phi'(x') = k' \cdot x' - \omega' t' = \phi(x) \]
\[ \Rightarrow \phi(x) = k_x \cdot x - \omega t = k_{n} x_{n} + \vec{k}_{\perp} \cdot \vec{x}_{\perp} - \frac{\omega}{c} x^0 \]

- parallel component along \( \vec{k} \) to \( \vec{u} \):
  \[ \vec{x}' = \vec{x} - \frac{\vec{u} \cdot \vec{x}}{\vec{u} \cdot \vec{u}} \vec{u} \]
  \[ x_n' = \gamma (x_n + \beta x^0) \quad x^0 = \gamma (x^0 + \beta x_n) \]
  \[ x_{\perp}' = \frac{x_{\perp}}{1 - \beta^2} \]

\[ \Rightarrow \phi(x') = k_{n}' x_n' + \vec{k}_{\perp}' \cdot \vec{x}_{\perp}' - \frac{\omega'}{c} x^0 \]

Thus \( k^{\mu} = (k^0, \vec{k}) \) with \( k^0 = \frac{\omega}{c} \) a four vector.

\[ k_{n}' = \gamma (k_{n} - \beta k^0) \Rightarrow \quad \omega' = \gamma (\omega - \beta k_{n}) \]
\[ k_{n}' = \gamma (k_{n} - \beta k^0) \Rightarrow \quad k_{n}' = \gamma (k_{n} - \frac{\omega^2}{c^2} \omega) \]
\[ \vec{k}_{\perp}' = \vec{k}_{\perp} \]

\[ \beta \ll 1 \quad \gamma \approx 1 \quad \omega' = \omega - k_{n} \frac{\omega}{c} \]

Note: Even for very moving \( v \) to observer there is a Doppler shift in \( \omega \)

\[ \frac{1}{k_{n}} = \frac{0}{v} \quad \Rightarrow \quad \omega' = \gamma \omega = \left(1 - \frac{v^2}{2c^2}\right) \omega \]

Generally \( \tan \theta = \frac{1}{k_{n}} \) changes
Other four vectors

- Proper velocity: First note that \( \frac{dx^\mu}{dt} \) is not a four-vector since it transforms under Lorentz. However if we use the proper time (i.e., time in the particle's own rest frame): \( d\tau \)

\[
U^\nu = \gamma(u) \frac{dx^\nu}{d\tau} \quad \text{Now} \quad d\tau = \frac{dt}{\gamma(u)} \quad \text{where} \quad u = \text{speed in lab frame}
\]

\[
U^\nu = \gamma(u) \frac{dx^\nu}{dt} = (\gamma(u)c, \gamma(u)\vec{u})
\]

- Four momentum Relativistic invariant

\[
U^\nu U_\nu = \gamma(u)^2 c^2 - \gamma(u)^2 U^2 = \gamma(u)^2 (c^2 - u^2)
\]

\[
= \frac{1}{1 - u^2/c^2} (c^2 - u^2) = c^2
\]

- Propa Four-momentum

\[
p^\nu = m U^\nu = \gamma(u) m \frac{dx^\nu}{d\tau} = (\gamma(u)mc, \gamma(u)\vec{u})
\]

Empirical fact: All components of \( p^\nu \) are conserved

\[
p^0 = \gamma(u)mc = E/C \quad \text{relativistic energy/c}
\]

Limit \( u \ll c \) \( \gamma(u) \approx 1 + \frac{u^2}{2c^2} \Rightarrow E^0 = mc^2 + \frac{1}{2} mu^2 \)

Relativistic invariant \( p^\nu p_\nu = (mc)^2 = (E^0)^2 - |p|^2 \)
Four-current density

Charge conservation: True in all reference frames
\[ \nabla \cdot \vec{J} + \frac{2}{\partial t} = 0 \Rightarrow \nabla' \cdot \vec{J}' + \frac{2}{\partial t'} = 0 \]

\[ \frac{2}{\partial x_{ii}} J_{ii} + \nabla \cdot \vec{J} + \frac{2}{\partial t} = 0 \]

Chain rule
\[
\begin{align*}
\frac{\partial}{\partial x_{ii}} &= \frac{\partial x'_{ii}}{\partial x_{ii}} \frac{\partial}{\partial x'_{ii}} + \frac{\partial t'}{\partial x_{ii}} \frac{\partial}{\partial t'} = \gamma \left( \frac{\partial x'_{ii}}{\partial t'} - \frac{v}{c^2} \frac{\partial}{\partial t'} \right) \\
\frac{\partial}{\partial t} &= \gamma \left( \frac{\partial}{\partial t'} - v \frac{\partial}{\partial x'_{ii}} \right) \\
\nabla' &= \nabla \gamma
\end{align*}
\]

\[ \Rightarrow \nabla \cdot \vec{J} + \frac{2}{\partial t} = \gamma \left( \frac{\partial x'_{ii}}{\partial t'} - \frac{v}{c^2} \frac{\partial}{\partial t'} \right) J_{ii} + \nabla \cdot \vec{J} + \gamma \left( \frac{\partial}{\partial t'} - v \frac{\partial}{\partial x'_{ii}} \right) \rho
\]

\[ = \frac{2}{\partial x_{ii}} \left[ \gamma (J_{ii} - v \rho) \right] + \nabla' \cdot \vec{J} + \frac{2}{\partial t'} \left[ \gamma (\rho - \frac{v}{c^2} \rho) \right] \]

\[ \therefore \quad \vec{J}' = (c \rho, \vec{J}) \text{ is a Four vector} \]

\[ \vec{J} = 0 \text{ in S frame} \Rightarrow \rho' = \gamma \rho \text{ is S'} \]

Physically: Length contraction

\[ \begin{array}{c}
\text{Original} \quad \rightarrow \quad \text{contracted} \\
\text{Length} \quad \text{contraction} \quad \frac{L'}{L} = \frac{1}{\gamma}
\end{array} \]
Minkowski Force: \[ K^\mu = \frac{dp^\mu}{dt} = m \frac{d^2x^\mu}{dt^2} \]

"Kinematic Force" \[ \vec{F} = \frac{dp}{dt}, \quad K^\mu = \gamma \frac{dp^\mu}{dt} \]

\[ \Rightarrow \quad K^\mu = (\frac{\gamma dp}{dt}, \gamma \vec{F}) \]

Aside \[ u^\mu u_\mu = c^2 \Rightarrow \frac{d(u^\mu u_\mu)}{dt} = 0 \]

\[ \Rightarrow \quad K^\mu u_\mu = 0 = \gamma^2 \frac{dE}{dt} - \gamma^2 \vec{F} \cdot \vec{u} \]

\[ \therefore \quad \frac{dE}{dt} = \vec{F} \cdot \vec{u} \quad \text{(rate at which force does work on particle)} \]

\[ \Rightarrow \quad K^\mu = (\gamma \vec{F} \cdot \vec{u}, \gamma \vec{F}) \]

Suppose we boost to the instantaneous rest frame of the particle

In lab frame: \[ K^\mu = (\gamma \vec{F} \cdot \vec{u}, \gamma \vec{F}) \]

If boost along \( \vec{F} \): \[ K_{\nu}' = \gamma (K_{\nu} + u K_{\nu} \cdot \vec{u}) \]

\[ \Rightarrow \quad K_{\nu}' = \gamma \vec{F}_{\nu}' = \gamma \vec{F}_{\nu} \]
\[ \Rightarrow \quad \mathbf{F}^\prime_1 = \mathbf{F}^\prime_{1'} \]

If Boost \( \perp \) to \( \mathbf{F}^\prime \)

\[ \Rightarrow \quad \mathbf{k}_1 = \mathbf{k}^\prime_1 \Rightarrow \quad \gamma(\mathbf{u}) \mathbf{F}_1 = \mathbf{F}^\prime_{1'} \]

These are the transformation rules on forces where \( \mathbf{F} \) is the instantaneous force on the particle in its own rest frame.